Options - Introduction.

Options are contracts which allow 1 party to purchase a security at prearranged price and time(s).

Unlike a forward, there is no obligation to purchase a stock, thus the value of an option is never negative.

Options com in variety of flavors:

(i) Call options: Contract ensures holder right to purchase stock at prearranged price X.

(ii) Put options: Contract ensures holder right to sell stock at prearranged price X.

(iii) European Options: Option must be exercized at specific expiry date

(iv) American Options: Option may be excersized at any time before expiry date $0 < t < T$.

We would like to find value @ time 0. First find value @ exersize date:

Value of European Call.

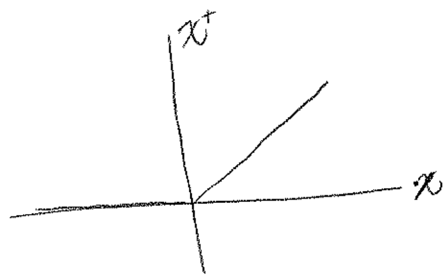
Option to purchase security ~~\$~~ for X at time T.

Value @ time T:

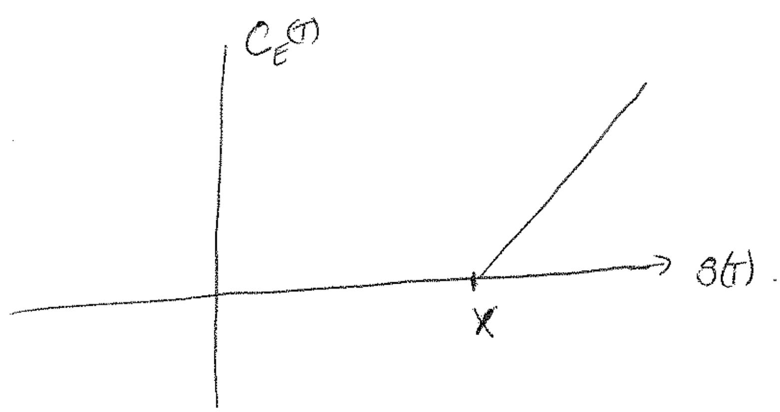
* if $S(T) > X$ purchase for X & sell for $S(T)$.
* if $X \geq S(T)$ do nothing collect \$0.

Let x^+ be notation defining:

$$x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$



Value of call \$ at time T, $C_E(T) = (S(T) - X)^+$



European Put,

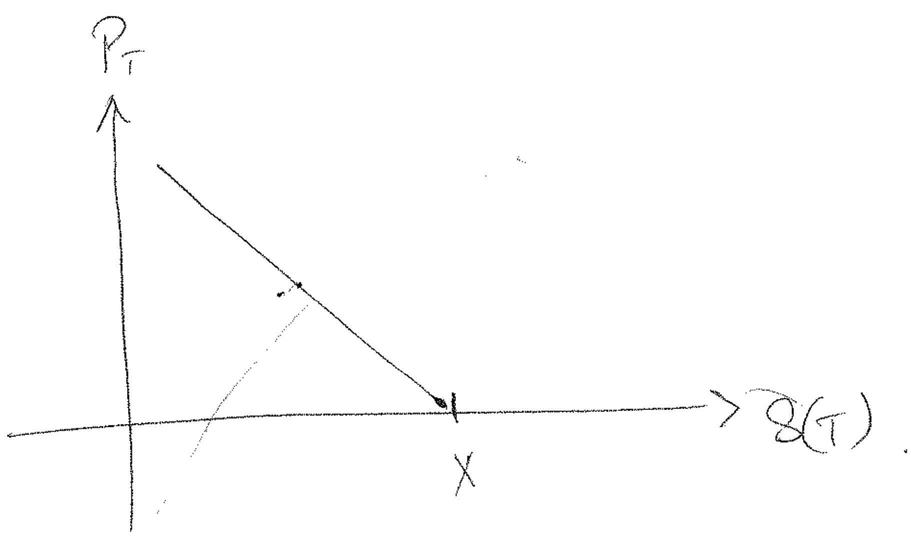
Option to sell security for X at time T,

Value * if $X > S(T)$ purchase for $S(T)$ + sell for X

* if $X \le S(T)$ do not excersise option.

Value of put:

$$P_T = (X - S(T))^+$$



Without specific assumption on probability space Ω

& variables $S(t)$ we cannot obtain precise value of the

call @ time $t = 0$.

We can determine upper & lower bounds of the
Value at time zero however.

Let us create portfolio w/ same payoff as Forward.

$\therefore$ Long Call & Short put portfolio,

Payoff at time T is,

$$V(T) = (S(T) - X)^+ - (X - S(T))^+$$

$$\therefore \quad S(T) \geq X \Rightarrow V(T) = S(T) - X$$

$$S(T) < X \Rightarrow V(T) = -(X - S(T))$$

$$\therefore \quad V(T) = S(T) - X.$$

$\therefore$ value of portfolio at time T is the same as
value of Forward @ maturity.

$\therefore$ Value of $C_E(0) - P_E(0)$ is the same as
value of forward contract to purchase
stock at price X at time T.

$\therefore$ Value of $C_E - P_E$ @ time 0:

Security w/o ~~derivative~~ dividend

Value of contract is $S(0) - X\,B(0,T) =: F_X(0,T)\,B(0,T)$

Security w/ dividend

$$S(0) - \sum \delta_i\, B(0,\tau_i) - X\,B(0,T) =: F_X(0,T)\,B(0,T)$$

Security w/ continuous dividend:

$$S(0)\,e^{-r_\delta T} - X\,B(0,T) =: F_X(0,T)\,B(0,T)$$

$\therefore$ In any case we write

$$C_E - P_E = F_X(0,T)\,B(0,T).$$

$$= (F(0,T) - X)\,B(0,T)$$

~~e~~ This equation:

$$\cancel{e}\ C_E - P_E = F_X(0,T)\, B(0,T)$$

is European Put-Call parity Equation.

Notice: there is "Paradox"

Suppose S is security at $\mathbb{E}(S(T)) >> X$ then

* Buying call @ X is attractive because we get expected profit $\mathbb{E}(S(T)) - X$.

* Buying Put option @ X is not attractive because we rarely have profit $(X - \mathbb{E}(S(T)))^+ = 0$

* However from put call parity

$$C_E - P_E = F_X(0,T)\, B(0,T)$$

if C_E increases then P_E increases

$\therefore$ $\mathbb{E}\, S(T)$ increases implies P_E increases $\lightning$.

#

Actually $C_E + P_E$ increases if <u>variance</u> of $S(T)$ increases. $C_E + P_E$ Do <u>Not</u> depend on expected returns

BOUNDS ON EUROPEAN Option prices.

Suppose $C_E \geq \left(F_X(0,T) + X\right)B(0,T)$

* Short option + buy stock
  (if continuous dividend ~~to~~ buy $e^{-\delta T}$ share)

* Invest difference in bond

$\therefore$ Profit (no dividend)

$$0 < \left\{C_E - S(0)\right\}\frac{1}{B(0,T)} + \min\left(S(T), X\right) \underbrace{\leq 0}_{\text{No arbitrage}}$$

$\lightning$.

$\therefore$ Profit (div.)

$$0 < \left\{C_E - \left(S(0) - \sum \delta_i B(0,\tau_i)\right)\right\}\frac{1}{B(0,T)} + \min\left(X, S(T)\right) \underbrace{\leq 0}_{\lightning}$$

$\therefore$ Profit (continuous div)

$$0 < \left\{C_E - S(0)\, e^{-\delta T}\right\}\frac{1}{B(0,T)} + \min\left(X, S(T)\right) \underbrace{\leq 0}_{\lightning}$$

$\therefore \quad C_E < \left(F_X + X\right)B(0,T)$

ON THE Other hand:

Put call parity:

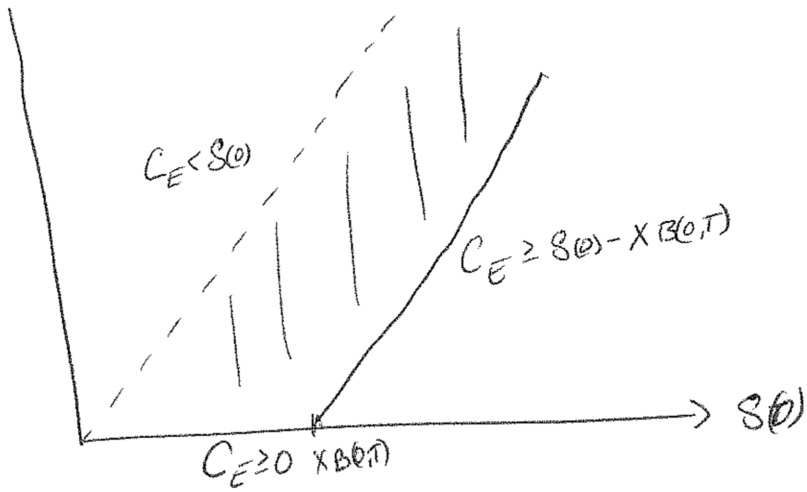
$$C_E - P_E = F_x(0,T)\, B(0,T)$$

$$C_E = F_x(0,T)\, B(0,T) + P_E \geq F_x(0,T)\, B(0,T)$$

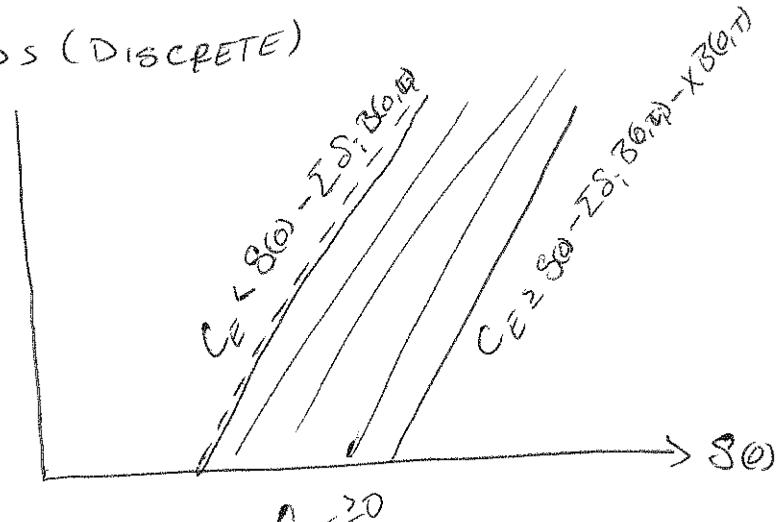
$\therefore$ we have BOUNDS.
for price of contract @ time 0

$$\left(B(0,T)\, F_x(0,T)\right)^{+} \leq C_E(0) < \left(F_x(0,T) + X\right) B(0,T)$$

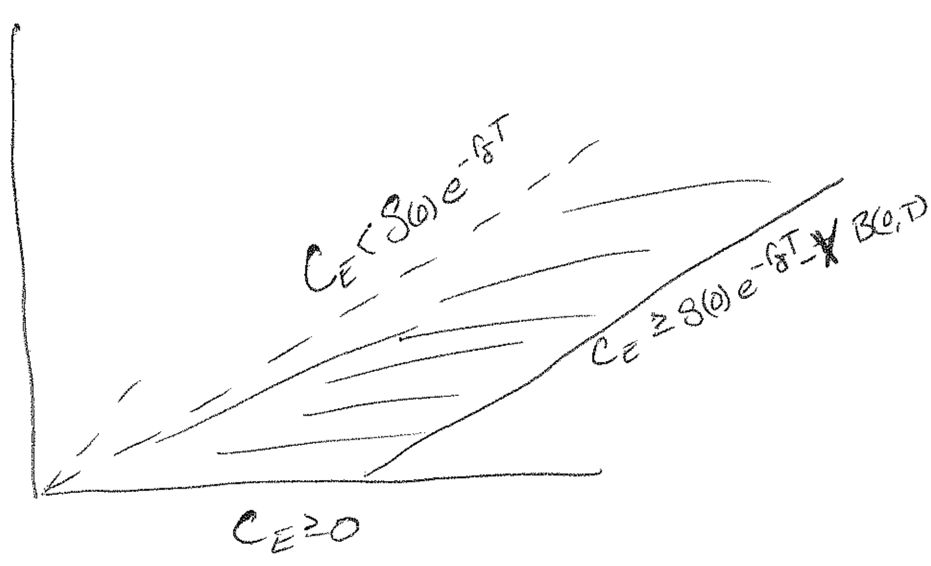
No Dividend:



~~D~~ DIVIDENDS (DISCRETE)



DIVIDENDS (CONTINUOUS)



BOUNDS FOR EUROPEAN PUT Options.

PC parity

$$C_E - P_E = F_X(0,T)\,B(0,T)$$

(i) $\Rightarrow 0 \le C_E = F_X(0,T)\,B(0,T) + P_E$

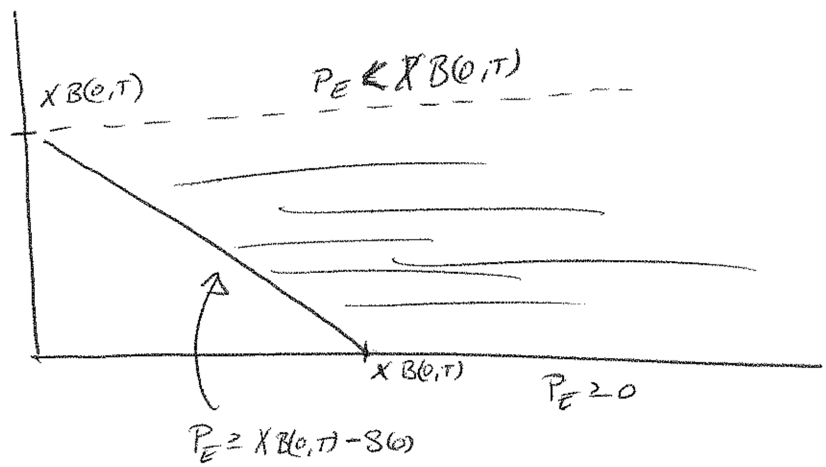
Now $P_E + F_X(0,T)\,B(0,T) = C_E < (F_X(0,T) + X)\,B(0,T)$

$$\Rightarrow P_E < X\,B(0,T)$$

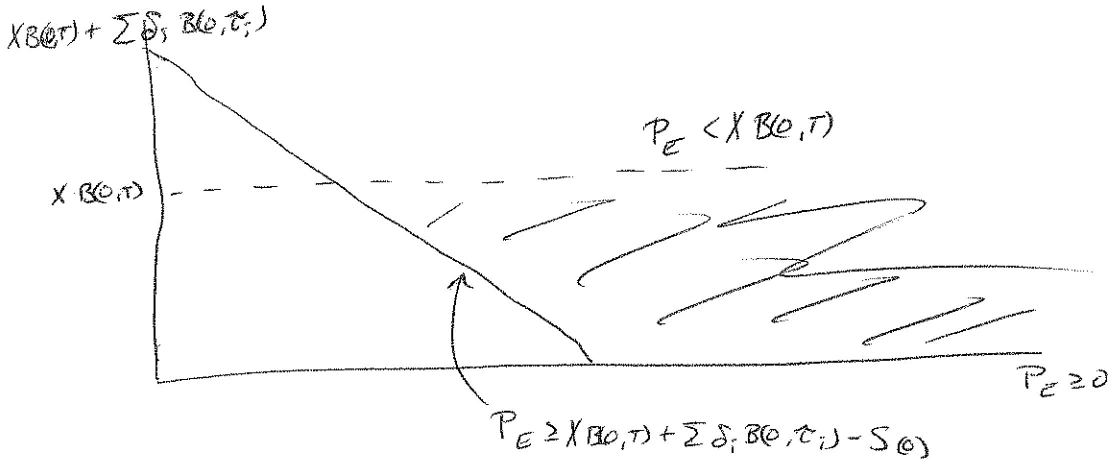
$$\therefore \quad -F_X(0,T)\,B(0,T) \le P_E < X\,B(0,T)$$

BOUNDS FOR PUT:

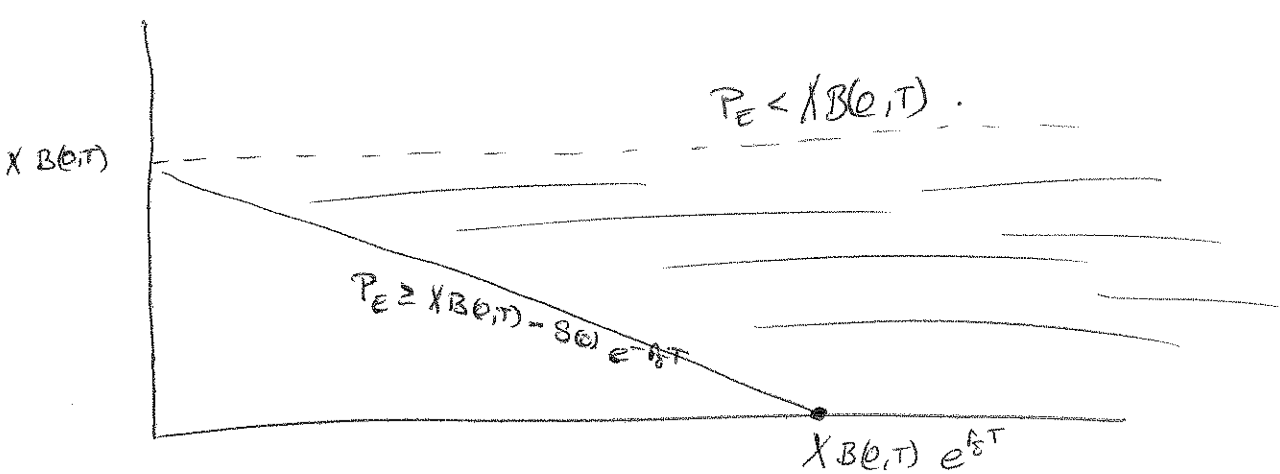
NO DIVIDEND:



DIVIDEND (DISCRETE)



DIVIDEND (CONTINUOUS)



EUROPEAN OPTION

Dependence on parameters ...

CALL STRIKE PRICE $\quad C_{E;X}(T) = (S(T) - X)^+$

$$X' < X'' \rightarrow \quad C_{E X'}(T) \geq C_{E X''}(T)$$

$$\Rightarrow \quad C_{E;X'}(0) \geq C_{E;X''}(0)$$

(*i)

PUT STRIKE PRICE

$$P_{E;X}(T) = (X - S(T))^+$$

$$X' < X'' \rightarrow$$

$$P_{E;X''}(T) \geq P_{E,X'}(T)$$

$$\hookrightarrow P_{E X''}(0) \geq P_{E X'}(0) .$$

(*ii)

IN FACT WE CAN DO BETTER.

CONSIDER ~~the~~ PUT CALL PARITY EQS:

$$C_{EX} - P_{EX} = F_X B(0,T) = F(0,T) B(0,T) - X B(0,T)$$

$$C_{EX'} - P_{EX'} = F(0,T) B(0,T) - X' B(0,T)$$

$$C_{EX''} - P_{EX''} = F(0,T) B(0,T) - X'' B(0,T)$$

taking difference $X' < X''$

$$\{C_{EX'} - C_{EX''}\} + \{P_{EX''} - P_{EX'}\} = (X'' - X') B(0,T)$$

$$\therefore \quad C_{EX'} - C_{EX''} \le (X'' - X') B(0,T)$$

$$\hookrightarrow \quad C_{EX''} \le C_{EX'} \le C_{EX''} + (X''-X') B(0,T)$$

$$(*i) \quad C_{EX'} - (X''-X') B(0,T) \le C_{EX''} \le C_{EX'}$$

$$\therefore \quad P_{EX''} - P_{EX'} \le (X'' - X') B(0,T)$$

$$\hookrightarrow \quad P_{EX'} \le P_{EX''} \le P_{EX'} + (X'' - X') B(0,T).$$

(*ii)

The Call & Put are actually convex in the strike price.

SUPPOSE $C_{EX} > \alpha\, C_{EX'} + (1-\alpha)\, C_{EX''}$

where $X'' > X'$ & $X = \alpha X' + (1-\alpha) X''$.

Write & sell contract with strike X

& buy contracts – α times contract w/ strike X'

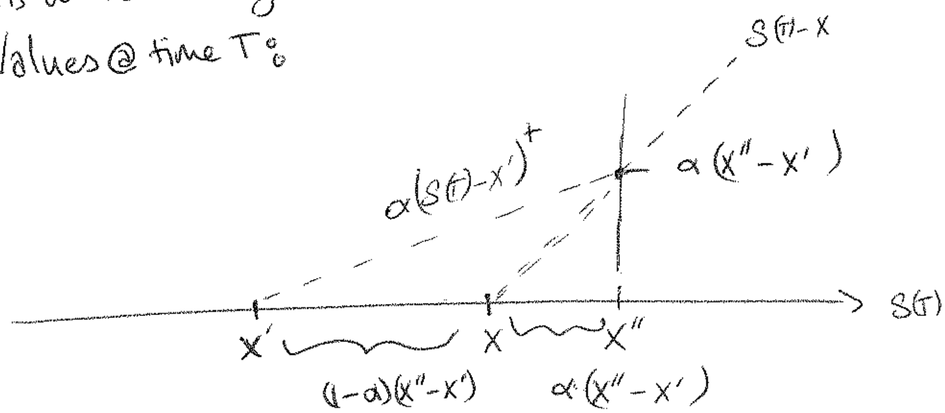
$(1-\alpha)$ times contract w/ strike X''

At expiry we gain $-(S(T) - X)^+$ (paying call)

and $\alpha\,(S(T) - X')^+ + (1-\alpha)\,(S(T) - X'')^+$

This is never negative:

Values @ time T:



when $S(T) \geq X''$

$\alpha\,(S(T)-X')^+ + (1-\alpha)\,(S(T)-X'')^+$

increases at rate 1.

$\therefore$ CONVEXITY

C_E & P_E are convex functions in the strike price

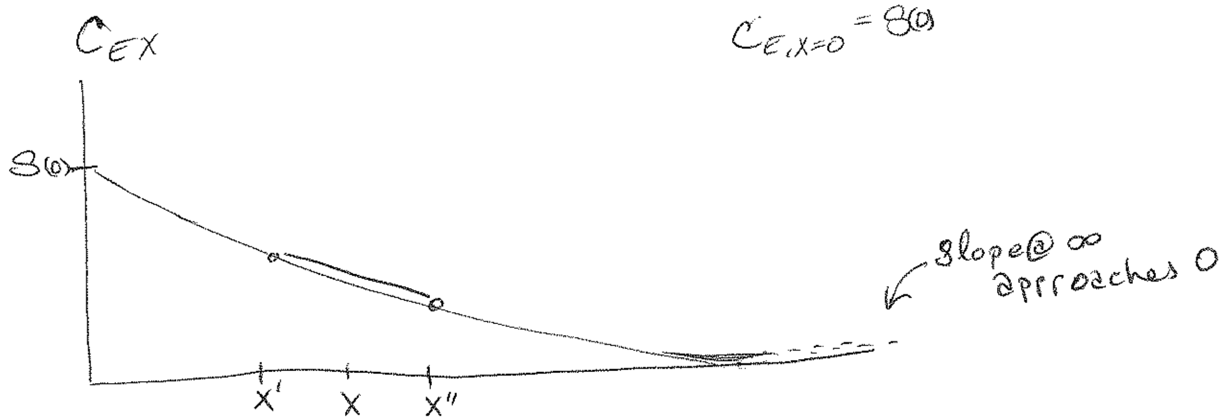
$$\left.\begin{aligned} C_{EX} &\le \alpha C_{EX'} + (1-\alpha) C_{EX''} \\ P_{EX} &\le \alpha P_{EX'} + (1-\alpha) P_{EX''} \end{aligned}\right\} \text{ for } X = \alpha X' + (1-\alpha) X''$$

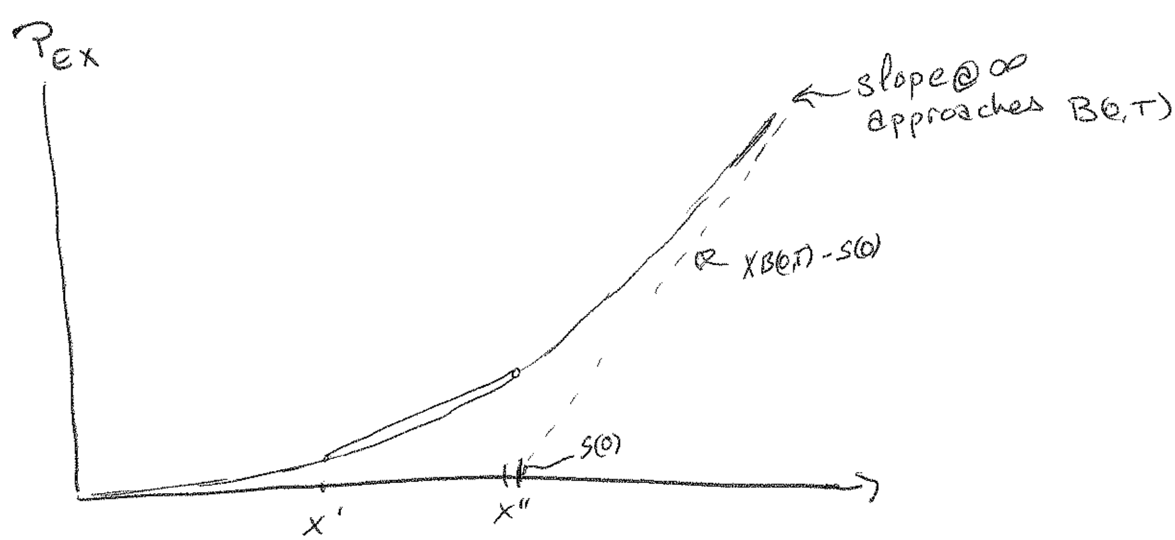
By PC parity we must have

$$\frac{d}{dX}\left(P_{EX} - C_{EX}\right) = B(0,T)$$

$$\& \; -P_{EX=0} + C_{EX=0} = S(0)$$

$$\& \; P_{E(X=0)} = 0$$
$$C_{E,X=0} = S(0)$$





<u>Fact</u> Similar bounds for C_E + P_E interms of share price S.

$\therefore$ $S' < S''$ time zero security prices:

$$C_{ES'} \leq C_{ES''} \leq C_{ES'} + (S'' - S') \quad \therefore \text{Increasing wrt } S(0)$$

$$P_{ES'} \geq P_{ES''} \geq P_{ES'} - (S'' - S') \quad \therefore \text{Decreasing wrt } S(0)$$

+ convex:

$$S = \alpha S' + (1-\alpha) S''$$

$$C_{ES} \leq \alpha C_{ES'} + (1-\alpha) C_{ES''}$$

$$P_{ES} \leq \alpha P_{ES'} + (1-\alpha) P_{ES''}$$

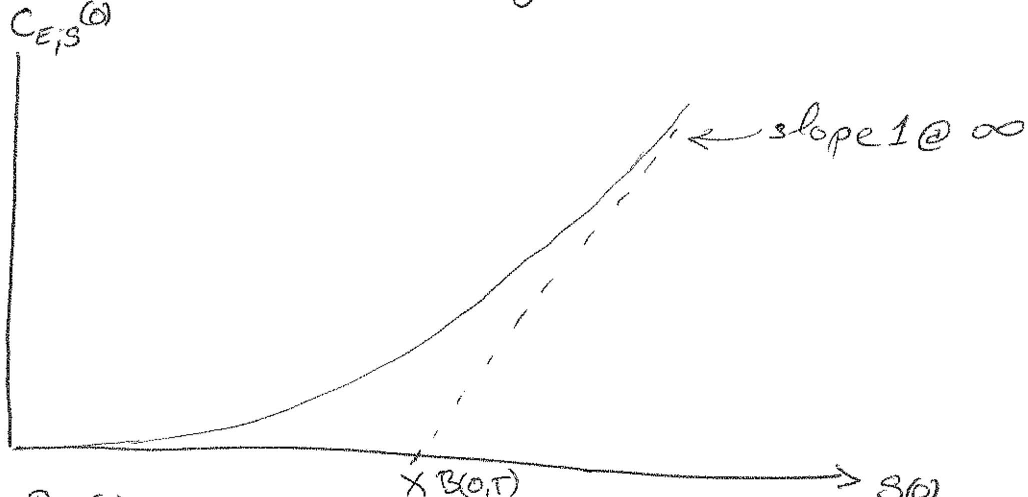
@ $S_0 = 0$

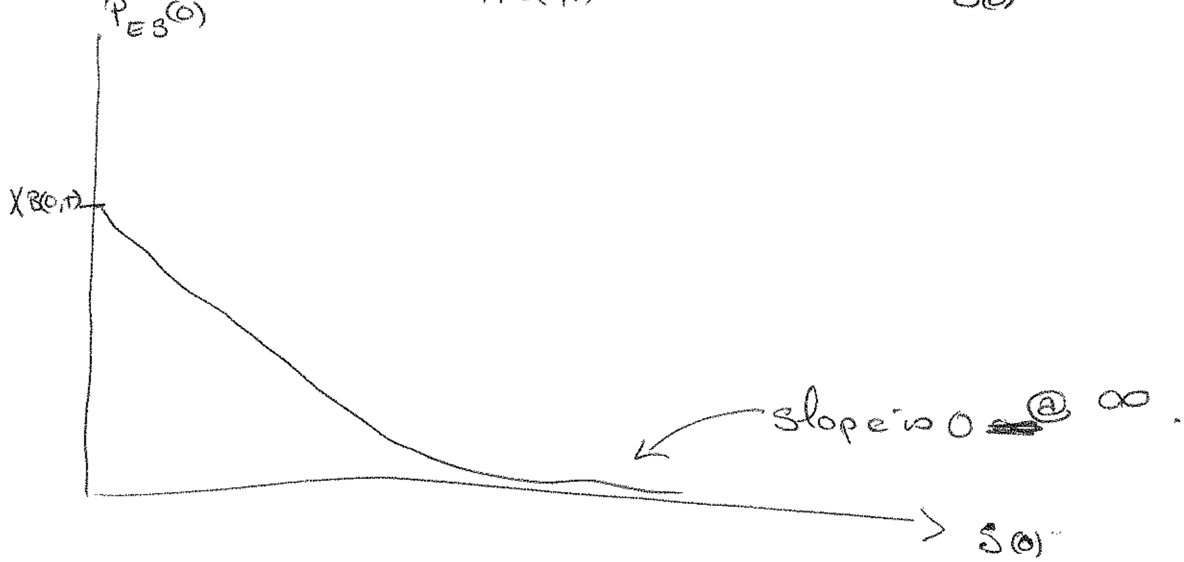
From bounds we have,

$$C_{E0} = 0$$
$$P_{E0} = XB(0,T)$$

Put is convex decreasing

Call is convex increasing...





Intrinsic value

INTRINSIC Value of option is the valere determined by excercise.

At time T ≡ expiry intrinsic value = exercise value.

$$C_E = (S(T) - X)^+$$

$$P_E = (X - S(T))^+$$

For time $t < T$ value is determined by same formula:

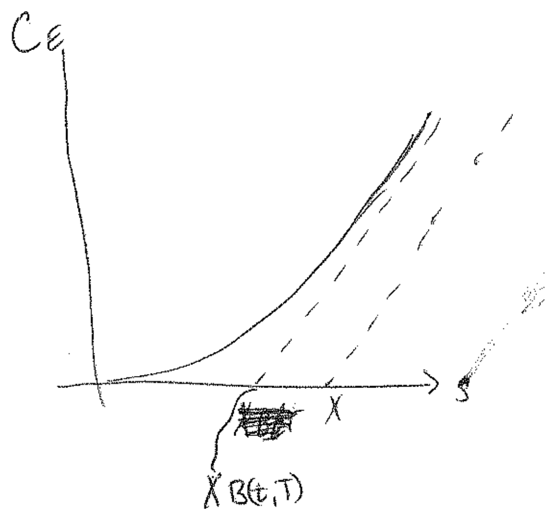
$$C^*_{E,t} = (S(t) - X)^+$$

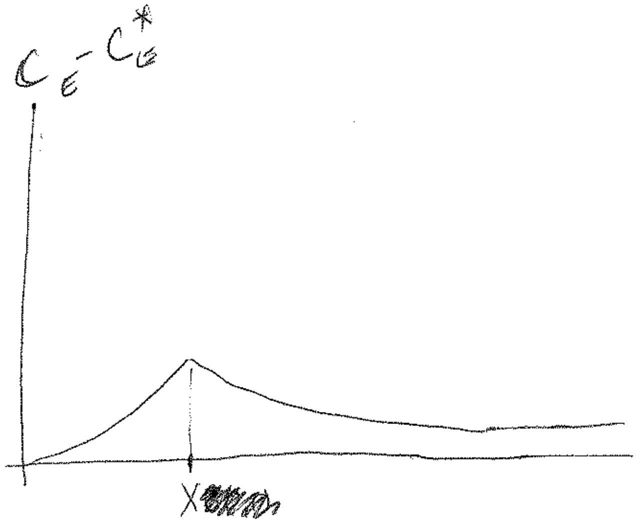
$$P^*_{E,t} = (X - S(t))^+$$

Time value is diff between intrinsic value + price

$$C_{Et} - C^*_{Et} \geq 0$$

$$P_{Et} - P^*_{Et}$$





∴ Value of Call is dropping wrt to time.

American Options.

* Call option (American) C_A

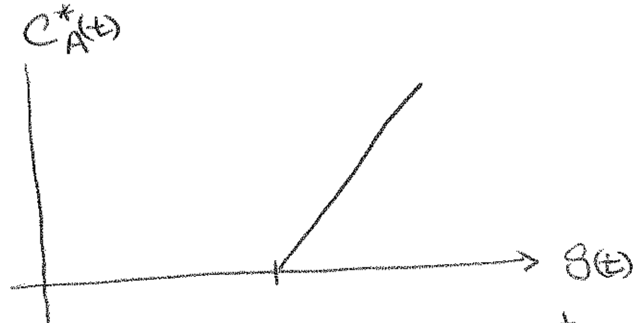
Contract allows you to purchase security
at any time t, $0<t<T$ for the strikeprice X.

* Put option (American)

Contract allows ~~you~~ holder to <u>sell</u> security
at any time t, $0<t<T$ for the strike price.

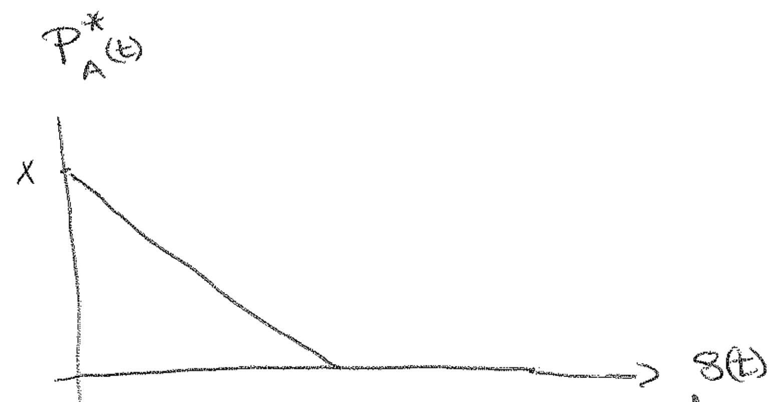
Intrinsic value: (at exercise).

Call



$$C^*_A(t) = (S(t) - X)^+$$

Put



$$P^*_A(t) = (X - S(t))^+$$

Clearly: $C_A(t) \geq C_E(t)$

$P_A(t) \geq P_E(t)$

Since American options allow more freedom than Euro options.

To obtain PC parity, ineq. Assume constant interest. $B(t,T) = e^{-r(T-t)}$

I write + sell call. Buy put + buy shared security.

* @ time $t=0$ \$ $C_A - P_A - S(0)$

* Exercise time $\underline{0<\tau<T}$

  $\underline{\text{Sell share for } X.}$

  We have put $P_A(\tau) \geq 0$ +

$$\$ \quad X + e^{r\tau}(C_A - P_A - S(0)) \leq 0$$

$$C_A - P_A \leq S(0) - e^{-\tau r} X$$

But we cannot control τ except $\tau < T$

$$\therefore \; C_A - P_A \leq S(0) - e^{-rT} X.$$

II Write & sell put, short share
Buy Call

* @ time $t=0$: \$ $S(0) + P_A - C_A$

* Put is exercized @ $\underline{0 < \tau < T}$

Buy Share for X,

then we have $C_A(\tau)$ and

$$\left(S(0) + P_A - C_A\right)e^{r\tau} - X \le 0$$

$$P_A - C_A \le e^{-r\tau}X - S(0)$$

But we cannot control τ,

$$\therefore P_A - C_A \le X - S(0)$$

$\therefore$

$$S(0) - Xe^{-rT} \ge C_A - P_A \ge S(0) - X$$

The value of the American call is the value of the European Call $\forall \ 0<t<T$

$$C_A(t) = C_E(t).$$

This follows from the fact:

One should __never__ exercise the american call early.

INFACT, the statement extends to any option $\mathcal{O}$ which has ~~payoff~~ Intrinsic value 0 if $S(t)=0$ + is convex in $S(t)$.

$$\begin{cases} \mathcal{O}^*(S(t)) \text{ is convex increasing in } S(t) \\ + \ \mathcal{O}^*(S(t)=0) = 0. \end{cases}$$

$\therefore$ If you are holding american option and you want to get rid of it, you do not exercise it before maturity, you would sell it on the market.

Consider Portfolio:

$*$ $t=0$ $\begin{cases} \text{Short american call} \\ \text{Long american put.} \\ \text{Balance @ } t=0 : \quad \$\left(C_A^{(0)} - C_E^{(0)}\right). \end{cases}$

$*$ $0<\tau<T$ $\begin{cases} \text{Suppose call is exercized @ time } \underline{0<\tau<T} \\ \text{Short share \& collect } X \text{ @ } \tau. \quad \$X \end{cases}$

$*$ $t=T$ $\begin{cases} \text{Euro call: buy share @ } X. \\ \text{Return to owner.} \end{cases}$

$$\$ \quad \left(C_A^{(0)} - C_E^{(0)}\right)\frac{1}{B(0,T)} + X\frac{1}{B(t,T)} - X \le 0$$

$$\therefore \quad C_A^{(0)} \le C_E^{(0)} + \left\{X - X\frac{1}{B(t,T)}\right\} B(0,T)$$

Cannot control τ $\therefore$

$$C_A^{(0)} \le C_E^{(0)}$$

But of course $C_A \ge C_E$

$$\therefore \quad C_E^{(0)} = C_A^{(0)}.$$

SINCE $C_A = C_E$

Bounds for $C_A = C_E$

$$S(0) - X B(0,T) \le C_A \le S(0)$$

For Puts:

Value @ excercise:

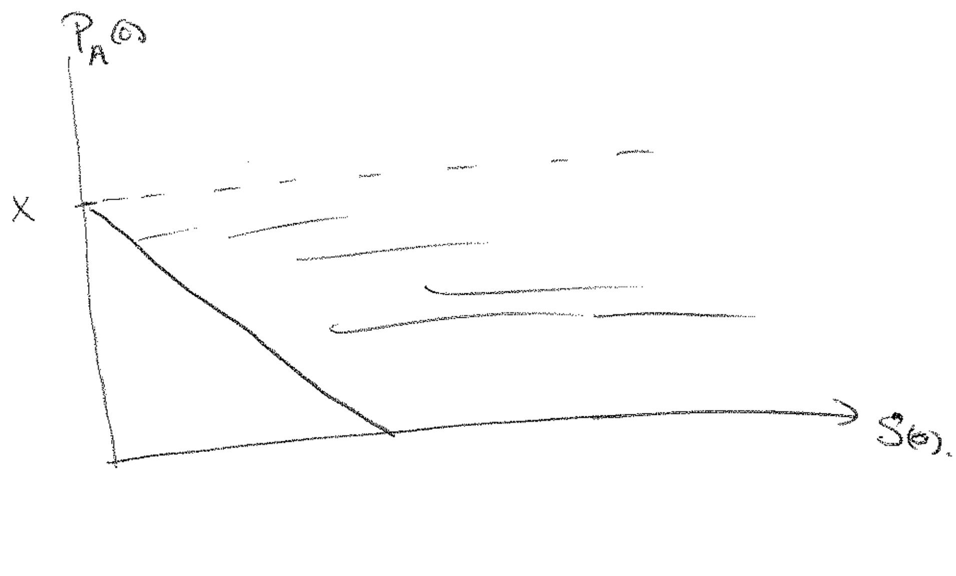
$$P_A^* = (X - S(t))^+$$

$$\therefore P_A^*(0) = (X - S(0))^+$$

$$\therefore P_A(0) \ge P_A^*(0) = (X - S(0))^+$$

On the other hand max payoff is $P_A^* \le X$

$$\therefore P_A(0) \le X$$



$\underline{Eg}$ Put Call parity:

$S(0) = \$36$

$r = .055$

$X = 37$

$C_A = \$2.03$

$\underline{T = 1/2}$

$S - X \le C_A - P_A \le S - Xe^{-r\frac{1}{2}}$

$2.03 \le 2.03 - 36 + 37e^{-r\frac{1}{2}} \le P_A$

$P_A \le S - X + C_A = \underline{\underline{3.03}}$