INTRODUCTION to ARBITRAGE.

Consider n securities over 2 time steps.

$\mathbb{T} \equiv$ time, $\mathbb{T} = \{0,1\}$ $\quad S_i(t) \equiv$ value of i^{th} security @ time t.

$S_i(0)$ given, $i = 1, \ldots, n$.

$\Omega \equiv$ sample space, $|\Omega| = m$

$S_i(1)$ are Random variables, $i = 1, \ldots, n$.

$$S_j(1) : \Omega \to (0, \infty)$$
$$\omega_i \mapsto S_j^{(\omega_i)}(1).$$

$$K_j = \frac{S_j(1) - S_j(0)}{S_j(0)}, \quad K_j : \omega_i \longmapsto K_j^{(\omega_i)}$$

Let A be the matrix

$$A_{ij} = K_j^{(\omega_i)}$$

A is $n \times m$ matix

Let purchase @ time 0 be written as:

$$X = (X_1, \ldots, X_n) \in \mathbb{R}^n.$$

X_i = \$ amount of i^{th} security purchased.

(we do not require $X_1 + \cdots + X_n = 1$).

Outcomes are (in linear algebra terms)

$$K_X = XA \quad \text{(Row vector)}$$

As Random Variable:

$$K_X : \Omega \to (0, \infty)$$

$$\omega_i \mapsto K_X^{\omega_i} = x_1 K_1^{\omega_i} + \cdots + x_n K_n^{\omega_i}$$

For given probability distribution.

$$\mathbb{P}(\omega_i) = q_i, \quad q_i \in (0,1), \quad q_1 + \cdots + q_n = 1.$$

q is apriori distribution given w/model.

Expected return:

$$\mathbb{E}(K_X) = \sum_{i=1}^{m} q_i K_X^{(\omega_i)} = x^T A q$$

$$q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{pmatrix}$$

* We have not introduced interest for sake of ~~simp~~ simplicity.

If interest is an issue then proper return variable is

$$\tilde{K}_i = \frac{(1+r)^{-1} S_i(1) - S_i(0)}{S_i(0)}$$

– or –

$$\tilde{K}_i = \frac{K_i - r}{1+r}$$

We say there exists an Arbitrage oportunity if there is ~~a portfolio~~ a portfolio $x \in \mathbb{R}^n$, so that $K_x \geq 0$ and $\mathbb{P}(K_x > 0) > 0$.

Ie there is positive probability of profit and zero probability of loss.

The following is the Arbitrage dichotomy:

Exactly ~~Only~~ one of the two following possibilities occurs for the matrix A.

(i) there is a portfolio x

so that $(xA)_j \geq 0 \quad \forall j$

There is some j_0 so that $(xA)_{j_0} > 0$

(ie in every scenario j we do not lose money & in some scenario j_0 we gain money)

this is an arbitrage opportunity.

(ii) There is a positive probability vector $\underset{\sim}{q}$ so that $\sum_{j=1}^{n} A_{ij} q_j = 0 \quad \forall i$

ie $A\underset{\sim}{q} = 0$

Proof

$$\left.\begin{array}{l}(i) \Rightarrow \text{not}\,(ii) \\ \text{not}(i) \Rightarrow (ii)\end{array}\right\} \Leftrightarrow \{\text{not}\,(i) \text{ or } \text{not}(ii)\} \text{ and } \{(i) \text{ or } (ii)\}$$

(A) $\{(i) \Rightarrow \text{not}\,(ii)\}$.

If (i) is true

$$\begin{cases} (xA)_j \geq 0 \quad \forall j \\ \exists j_0 \text{ s.t. } (xA)_{j_0} > 0 \end{cases}$$

for q_h positive probability vector:

$$xAq_h = \sum_{ij} x_i A_{ij} q_h^j$$

$$= \sum_j \left(\sum_i A_{ij}\right) q_h^j$$

$$\geq \left(\sum_i A_{ij_0}\right) q_h^{j_0} > 0.$$

$$\Rightarrow \quad Aq_h \neq 0.$$

(B) not (i) $\Rightarrow$ (ii).

Let $R_A = \{xA) : x \in \mathbb{R}^n\} \subset \mathbb{R}^m$ be the rowspace of A.

(i) $\Rightarrow$ $\exists\, x \in \mathbb{R}^n$ st $xA \in Q_1 \equiv$ the first quadrant of $\mathbb{R}^m$.

$\therefore$ not (i) $\Rightarrow R_A \cap Q_1 = \{0\}$

ie R_A intersects the first quadrant only at the origin.

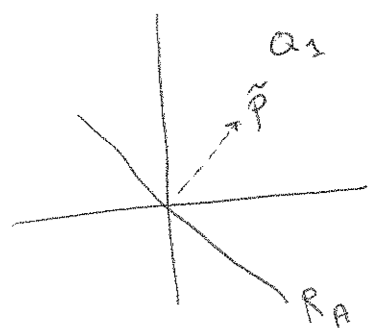
In particular $y \in R_A$ implies there is some j_1, j_2 st

$$y_{j_1} > 0 > y_{j_2}$$

$\therefore R_A$ can be contained in $m-1$ dimension subspace...

a plane, there is some $\tilde{p}$ st

$$R_A \subset \{y : \tilde{p}_1 y_1 + \cdots + \tilde{p}_m y_m = 0\}.$$



Extend R_A to R_A' an $m-1$ dim subspace not intersecting Q_1

then $\exists\, p$ st $\quad R_A \subset R_A' = \{y : p^T y = 0\}$.

But if p had some zero component $p_j = 0$ then

the vector $e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th}$ position would belong to R_A which is a contradiction. $\blacksquare$

EXAMPLE:

GIVEN the following system of bets determine whether or not there is an arbitrage oportunity:.
(It is permitted to take "negative bets").

Suppose MSU is playing UM,
there are the ~~following~~ 3 bets you can purchase for $1 with the following payoffs:

Ω	win	lose	draw
Bet 1	0	1	3/2
Bet 2	2	2	0
Bet 3	1/2	3/2	0
$\mathbb{P}$	p_1	p_2	p_3

Is there an arbitrage opportunity?

Payoff matrix:

$$A = \begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 1 & -1 \\ -1/2 & 1/2 & -1 \end{pmatrix}$$

Arbitrage $x = (x_1, x_2, x_3)$ st $\quad xA \geq 0 \quad$ & has at least 1 strictly positive term.

In this case $\det A \neq 0 \quad \therefore$
we can choose any payoff vector, say $\begin{matrix}1\\1\\1\end{matrix} = \mathbb{1}$.

And solve $\quad x = A^{-1}\mathbb{1}$.

GIVEN SAME EXAMPLE BUT SUPPOSE

Bet 3 is not avzilzble:

$$A = \begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 1 & -1 \end{pmatrix} \quad \therefore \quad \exists p \text{ st } Ap = 0 \ ?$$

~~find Row space:~~ Row Reduce

$$\cancel{R_A = Sp\left\{\begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 1 & -1 \end{pmatrix}\right\} = Sp\left\{\begin{pmatrix} -1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{pmatrix}\right\}}$$

$$\begin{pmatrix} -1 & 0 & 1/2 \\ 1 & 1 & -1 \end{pmatrix} p = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} -1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{pmatrix} p = \begin{matrix} y_1 \\ y_1 + y_2 \end{matrix}$$

want $y_1 = y_2 = 0$

$$-p_1 + p_3 \tfrac{1}{2} = 0$$

$$p_2 - \tfrac{1}{2} p_3 = 0$$

$$p_1 = \tfrac{1}{2} p_3$$

$$p_2 = \tfrac{1}{2} p_3.$$

$$\longrightarrow p = (2, 2, 1)$$

~~[illegible]~~ or $p = (2/5, 2/5, 1/5)$

$\exists p$ st $Ap = 0$

$\therefore$ there is no arbitrage oportunity.

Now consider 2 BETS Both costing \$10 w/ payoff

	win	Lose	draw.
Bet 1	17	0	16
Bet 2	18	16	7
π	q_1	q_2	q_3.

Normalize:

$$A = \begin{pmatrix} .7 & -1 & .6 \\ .8 & .6 & -.3 \end{pmatrix}$$

$\exists p$ st $Ap = 0$?

reduce A:

$$A' = \begin{pmatrix} 2.3 & .2 & 0 \\ .8 & .6 & -.3 \end{pmatrix} \begin{array}{l} r_1 + 2r_2 \\ r_2 . \end{array}$$

If $A'p = 0$ then $2.3p_1 + 1.1 p_2 = 0$

$$p_1 = -\frac{.2}{2.3} p_2 \Rightarrow p_1 = p_2 = 0$$

But then $-.3p_3 = 0 \Rightarrow p_3 = 0$

So $Ap = 0 \Rightarrow p = 0$ $\therefore$ Arbitrage exists... find arbitrage

$$\begin{pmatrix} .7 & .8 \\ -1 & .6 \\ .6 & -.3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad \begin{pmatrix} .7 & .8 \\ .2 & 0 \\ .6 & -.3 \end{pmatrix} \begin{array}{ll} x_1 & y_1 \\ x_2 & y_2 + 2y_3 \\ & y_3 \end{array}$$

$$2x_1 = x_3$$