Math 458 - Practice Problems for Quiz # 4 - Fall 2023

1. Let $S(t)$ be the price of a stock at time t, modeled via

$$dS(t) = 0.1\,S(t)\,dt + 0.2\,S(t)\,dW(t).$$

Calculate the probability p that after two years, the stock price is larger than its initial price.

Solution: The equation is geometric Brownian motion with solution $S(t) = S(0)e^{(\alpha-\sigma^2/2)t+\sigma W(t)}$ to $dS = \alpha S dt + \sigma S dW(t)$. For the given parameters we have the stock price process $S(t) = S(0)e^{(0.1-0.2^2/2)t+0.2W(t)} = S(0)e^{0.08t+0.2W(t)}$. It follows that

$$\begin{aligned}
\mathbb{P}(S(2) > S(0)) &= \mathbb{P}\left(S(0)e^{0.08(2)+0.2\sqrt{2}Z} > S(0)\right) = \mathbb{P}\left(e^{0.08(2)+0.2\sqrt{2}Z} > 1\right),\\
&= \mathbb{P}(0.16 + 0.2\sqrt{2}Z > 0) = \mathbb{P}\left(Z > -\frac{0.16}{0.2\sqrt{2}}\right) = \mathbb{P}(Z > -0.565685),\\
&= 1 - \mathbb{P}(Z \leq -0.565685) = 1 - 0.285804 = \boxed{71.4196\%.}
\end{aligned}$$

2. If $W(t)$ is a standard Brownian motion and

$$X(t) = W(t)^3 + c\,t\,W(t)$$

is a martingale, find c.

Solution: Setting $f(W,t) = W^3 + ctW$, we have $f_t = cW$, $f_W = 3W^2 + c\,t$, & $f_{WW} = 6W$ so Ito's formula yields

$$\begin{aligned}
dX &= f_W\,dW + \left(f_t + \frac{1}{2}f_{WW}\right)dt = \left(3W^2 + c\,t\right)dW + \left(c\,W + \frac{1}{2}\cdot 6W\right)dt,\\
&= (3-c)W\,dt + \left(3W^2 + c\,t\right)dW.
\end{aligned}$$

Using the given hint, it follows that the Itô process is a martingale if and only if $3 - c = 0$ or $\boxed{c = 3.}$

3. Let $W(t)$ be a standard Brownian motion. Compute the quadratic variation of

$$X(t) = 3 + 4W(t)$$

from $t = 0$ to $t = 2$.

Solution: Since $dX = 4dW$ and $(dW)^2 = dt$, it follows that the quadratic variation is

$$\int_0^2 (dX)^2 = \int_0^2 (4\,dW)^2 = \int_0^2 16(dW)^2 = 16\int_0^2 dt = \boxed{32.}$$

4. $\{X(t)\}_{t\geq 0}$ follows arithmetic Brownian motion such that $X(45) = 41$. The drift factor of this Brownian motion is 0.153, and the volatility is 0.98. What is the probability that $X(61) < 50$?

<u>*Solution:*</u> We are given $X(t) = 0.153\,t + 0.98\,W(t)$, where $W(t)$ is standard Brownian motion. Since $X(45) = 41$, it follows that

$$\begin{aligned} X(61) &= X(61) - X(45) + X(45) = 0.153 \cdot (61 - 45) + 0.98\,(W(61) - W(45)) + 41, \\ &= 43.448 + 0.98\,(W(61) - W(45))\,; \quad W(61) - W(45) \sim N(0; 16). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{P}\,(X(61) < 50) &= \mathbb{P}\,(43.448 + 0.98\,(W(61) - W(45)) < 50) = \mathbb{P}\left(\sqrt{16}\,Z < \frac{50 - 43.448}{0.98}\right), \\ &= \mathbb{P}(Z < 1.67143) \simeq \boxed{95.2682\%.} \end{aligned}$$

5. For $W(t)$ standard Brownian motion, show that

$$\mathbb{P}(W(1) \leq 0 \quad \text{and} \quad W(2) \leq 0) = \frac{3}{8}.$$

<u>*Solution:*</u> Let $Z_1 = W(1) - W(0) = W(1)$ and $Z_2 = W(2) - W(1)$ so that $Z_{1,\,2}$ are independent standard normal random variables with joint PDF

$$n(z_1,\, z_2) = n(z_1)n(z_2) = \frac{e^{-z_1^2/2}}{\sqrt{2\pi}} \cdot \frac{e^{-z_2^2/2}}{\sqrt{2\pi}}.$$

Now, using $n(x) = N'(x)$ where $N(x) = \mathbb{P}(Z \leq x)$, we find

$$\begin{aligned} \mathbb{P}\,(\{W(1) \leq 0 \wedge W(2) \leq 0\}) &= \mathbb{P}\,(\{W(1) \leq 0 \wedge W(2) - W(1) + W(1) \leq 0\})\,, \\ &= \mathbb{P}\,(\{Z_1 \leq 0 \quad \text{and} \quad Z_2 + Z_1 \leq 0\}) = \int_{-\infty}^{0} \left[\int_{-\infty}^{-z_1} \frac{e^{-z_1^2/2}}{\sqrt{2\pi}} \frac{e^{-z_2^2/2}}{\sqrt{2\pi}}\, dz_2\right] dz_1, \\ &= \int_{-\infty}^{0} \frac{e^{-z_1^2/2}}{\sqrt{2\pi}} \left[\int_{-\infty}^{-z_1} \frac{e^{-z_2^2/2}}{\sqrt{2\pi}}\, dz_2\right] dz_1 = \underbrace{\int_{-\infty}^{0} \frac{e^{-z_1^2/2}}{\sqrt{2\pi}} \cdot N(-z_1)\, dz_1}_{x = -z_1,\; dx = -dz_1}, \\ &= \int_{0}^{\infty} \underbrace{\frac{e^{-x^2/2}}{\sqrt{2\pi}}}_{= N'(x)} N(x)\, dx = \int_{0}^{\infty} N(x) N'(x)\, dx = \int_{0}^{\infty} \frac{1}{2} \frac{d}{dx} \left[N^2(x)\right] dx, \\ &= \frac{1}{2} \cdot N^2(x)\Big|_{x=0}^{x\to\infty} = \frac{1}{2} \cdot \left(1^2 - \left(\frac{1}{2}\right)^2\right) = \frac{1}{2} \cdot \frac{3}{4} = \boxed{\frac{3}{8}.} \end{aligned}$$

6. Solve the stochastic differential equation

$$dX(t) = \frac{b - X(t)}{1-t}\,dt + dW(t)$$

where $0 \le t < 1$ and $X(0) = a$. Here a, $b \in \mathbb{R}$ are arbitrary (but fixed) constants.

Solution: First note for $f(X,t) = X/(1-t)$ that Itô's formula gives

$$\begin{aligned} d\left(\frac{X(t)}{1-t}\right) &= \frac{X(t)}{(1-t)^2}\,dt + \frac{dX(t)}{1-t}, \\ &= \frac{1}{1-t}\left(dX(t) + \frac{X(t)}{1-t}\,dt\right). \end{aligned}$$

Further, the original SDE can be rewritten as

$$dX(t) + \frac{X(t)}{1-t}\,dt = \frac{b}{1-t}\,dt + dW(t)$$

so that

$$d\left(\frac{X(t)}{1-t}\right) = \frac{1}{1-t}\left(\frac{b}{1-t}\,dt + dW(t)\right) = \frac{b}{(1-t)^2}\,dt + \frac{dW(t)}{1-t}.$$

Integrating the previous expression we get

$$\frac{X(t)}{1-t} - \frac{X(0)}{1-0} = \int_0^t \frac{b}{(1-s)^2}\,ds + \int_0^t \frac{dW(s)}{1-s}.$$

It then follows from $X(0) = a$ that

$$\boxed{X(t) = a(1-t) + bt + (1-t)\int_0^t \frac{dW(s)}{1-s}.}$$