Math 458 - Practice Quiz # 2 - Solution

1. (1 point each) Please circle either T (true) or F (false) for each of the below statements. **Answers are in BOLD.**

A) T **F** $-C(K)$ is a convex function of the strike price K.

B) **T** F If $P(22) = 2$ and $P(23) = 4$ on a non-dividend paying stock, a no-arbitrage property for put prices is violated.

C) **T** F For $C^E(K)$, the arbitrage-free price of a call option with strike price K, satisfies $C^E(10) \leq C^E(12) + 2$.

D) **T** F The risk-neutral probability $\tilde{p}$ in a 1-period/1-year binomial asset pricing model for an asset with price S can be derived from the equation

$$\tilde{p} \cdot (Su) + (1 - \tilde{p}) \cdot (Sd) = Se^{(r-\delta)h}.$$

2. (6 points) It is known that for 1-year European put options, the arbitrage-free price $P^E(K)$ satisfies

$$P^E(40) = 4 \quad \text{and} \quad P^E(50) = 9.$$

A) (4 points) Use convexity to find an upper bound for the price of a European put option with strike price 44.

Solution: Let $K_1 = 40$, $K_2 = 44$, and $K_3 = 50$. Then

$$\lambda K_1 + (1 - \lambda)K_3 = 40\lambda + 50(1 - \lambda) = 50 - 10\lambda = 44 \quad \Rightarrow \quad \lambda = \frac{3}{5}.$$

Convexity then implies that

$$P^E(44) = P^E\left(\frac{3}{5} \cdot 40 + \frac{2}{5} \cdot 50\right) \leq \frac{3}{5} \cdot P^E(40) + \frac{2}{5} \cdot P^E(50) = \frac{3 \cdot 4 + 2 \cdot 9}{5} = 6$$

so that $\boxed{P^E(44) \leq 6.}$

B) (2 points) Find a lower bound for $P^E(44)$ that is greater than 0.

Solution: Monotonicity guarantees that

$$P^E(50) - P^E(44) = 9 - P^E(44) \leq (50 - 44) = 6 \quad \Rightarrow \quad P^E(44) \geq 9 - 6 = 3$$

so that $\boxed{P^E(44) \geq 3.}$

3. (4 points) Let the current price of a stock be 35. Let $C(S, K, T)$ and $P(S, K, T)$ be European calls and puts respectively on the stock with strike price K and expiration T. Show that

$$P(S, 35, T) - C(S, 30, T) \geq 30e^{-rT} - 35e^{-\delta T}.$$

Solution: First note from monotonicity we know that $P(S, 35, T) \geq P(S, 30, T)$. It then follows from PCP that

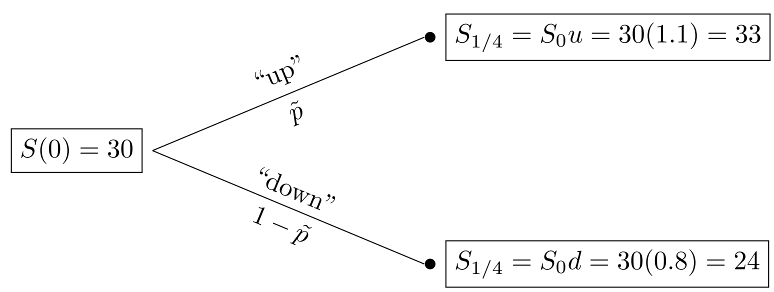
$$P(S, 35, T) - C(S, 30, T) \geq P(S, 30, T) - C(S, 30, T) = 30e^{-rT} - 35e^{-\delta T}$$

so therefore

$$\boxed{P(S, 35, T) - C(S, 30, T) \geq 30e^{-rT} - 35e^{-\delta T}.}$$

4. A non-dividend paying stock with current price \$30 either increases by 10% or decreases by 20% in three months. If the continuous annual interest rate is 4%, find the arbitrage-free price of a 3-month \$35-strike European put option. Use the 1-period binomial model.

Solution: The up/down factors are $u = 1.1$ and $d = 0.8$ every three months. The 1-period tree for S_t for $t = 0$ and $t = T = 1/4$ is thus



Now, the payoff of the European put option with strike \$35 is

$$\Lambda(S_{1/4}) = (35 - S_{1/4})^+ = \left\{ \begin{array}{ccc} (35-33)^+ = 2 & : & \text{"up case"} \\ (35-24)^+ = 11 & : & \text{"down case"} \end{array} \right\}.$$

Therefore, the replicating portfolio $\Pi = \{\Delta, B\}$ consisting of Δ shares of the stock and B invested at the risk-free rate $r = 4\%$, satisfies

$$\Delta \cdot 33 + Be^{0.04/4} = 2 \quad \text{and} \quad \Delta \cdot 24 + Be^{0.04/4} = 11$$

or

$$\Delta = \frac{2 - 11}{33 - 24} = -1 \quad \text{and} \quad B = \frac{11 - (-1)24}{e^{0.04/4}} = 35e^{-.01} \simeq 34.6517.$$

It follows that the arbitrage-free value of the replicating portfolio Π at $t = 0$ is

$$V_\Pi(0) = 30\Delta + B = 30(-1) + 34.6517 = \boxed{\$4.6517.}$$

<u>Alternative Method:</u> Using the risk-neutral probability

$$\tilde{p} = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.04/4} - 0.8}{1.1 - 0.8} \simeq 0.700167,$$

$V_{\Pi}(0)$ can be computed as

$$V_{\Pi}(0) = P^E = e^{-0.04/4}\tilde{E}\left[\Lambda(S_{1/4})\right] = e^{-0.01}\left(\tilde{p} \cdot 2 + (1-\tilde{p}) \cdot 11\right) \simeq \boxed{\$4.6517.}$$