Math 458 - Financial Math for Actuaries II - Practice Final Exam - Solution

1. Please circle either T (true) or F (false) for each of the below statements. Each correct answer is worth 1 point. There is no partial credit or penalty for guessing. **Answers are in BOLD.**

I) T **F** A short European put option with strike $K_1 = 20$ combined with a long European put option with strike $K_2 = 30$ forms a long bull spread.

II) T **F** If the continuous risk-free rate is always equal to the continuous dividend rate on a stock with price S_t at time t, then arbitrage opportunities do not exist for forward contracts.

III) **T** F It is impossible for an American call to have premium 20 at $t = 0$ when the initial stock price is 15.

IV) **T** F It is possible for $0 < d < u < 1$ in a risk-neutral binomial tree.

V) T **F** In a symmetric random walk M_k with $M_0 = 0$, it is possible for $M_{17} = 0$.

VI) T **F** It is possible, when $K = 4$ and $S(0) = 5$, for a European call option to have premium $C^E = 1$ while for the related up-and-in call, $C^{ui} = 0.32$.

VII) **T** F Within the BSM framework, θ for a long put option can be positive or negative.

VIII) **T** F The Δ-Γ-θ approximation can be used to approximate the European call premium difference $C(S_{t+h}) - C(S_t)$ if $S_{t+h} = S_t$ for $h > 0$. That is, the approximation is valid if the stock price does not change but time *does* change.

IX) T **F** Within the BSM framework, it is possible that $\Delta = 0$ for a European put.

X) T **F** On a Jarrow-Rudd binomial tree, geometrically averaged strike Asian European options are not path dependent.

2. Please circle either T (true) or F (false) for each of the below statements. All prices are no-arbitrage prices. **Answers are in BOLD.**

I) **T** F Combining a short straddle with strike price $K_2 = \$60$ with a long strangle with strikes $K_1 = 30$ and $K_3 = 90$ forms a symmetric butterfly spread.

II) **T** F If $r = \delta$ for a stock with price S_t, then the forward price $F_{0,T}$ is equal to the asset price S_0.

III) T **F** It is never advantageous to exercise an American call option early.

IV) T **F** The arithmetic Brownian motion

$$X(t) = 5 + 13t + 2W(t)$$

satisfies

$$\text{Var}(X(t+3) - X(t)) = 6.$$

V) T **F** In the Cox-Ross-Rubenstein binomial model, if $u = 1.2$ then $d = 0.8$.

VI) **T** F In a 5-period binomial tree with initial stock price $S_0 = \$100$, expiration $T = 1$ year, $u = 1.1$, and $d = 0.9$, the second highest value of S_1 is \$131.769.

VII) **T** F Given a risk-free rate $r > 0$ and continuous dividend rate $\delta > 0$, one way to compute the risk-neutral probability $\tilde{p}$ for the binomial model is to require that the risk-neutral expected value of the stock S_{t+1} is equal to the forward price of the stock. That is, solve

$$\tilde{E}[S_{t+1}] = \tilde{p}(uS_t) + (1 - \tilde{p})(dS_t) = S_t e^{(r-\delta)h}.$$

for $\tilde{p}$.

VIII) **T** F If the $C(S)$ is the price of a European call and you know that $C(40) = 2.7804$, $\Delta(40) = 0.5824$, and $\Gamma(40) = 0.0652$, then $C(40.75) \approx 3.2355$. Assume that all other problem parameters (including time) are held constant.)

IX) T **F** Because the Black-Scholes formula gives the arbitrage-free price of a call option in continuous time, you cannot use it to price a call on a stock where only discrete dividends are paid (e.g., owning a stock pays dividends only twice/year).

X) T **F** The fact that the "Greek" $\Gamma > 0$ is always true guarantees that C^E and $-P^E$ are convex.

3. You are given the following information on a non-dividend paying stock:

   - The stock price is 50.
   - The price of a 6-month European call option with an exercise price of 48 is 5.
   - The price of a 6-month European put option with an exercise price of 48 is 3.
   - The risk-free interest rate is constant 8%, compounded continuously.

   A) There is an arbitrage opportunity involving buying or selling one share of stock and buying or selling puts and calls. Construct an arbitrage portfolio Π for this scenario.

   Solution: Put-call parity (PCP) shows that for $C = 5$ the price of a put should be

   $$P = C + Ke^{-rT} - S(0) = 5 + 48e^{-(0.08)/2} - 50 \simeq 1.1179$$

   so that the put is *overpriced*. Alternatively, if $P = 3$ then PCP requires that

   $$C = P - Ke^{-rT} + S(0) = 3 - 48e^{-0.04} + 50 \simeq 6.882$$

   so that the call is *underpriced*. Therefore, arbitrage is achieved by shorting the put and buying the call, the net effect being a synthetic forward. The portfolio is

   $$\boxed{\Pi = \{-1, 1, -1, 48\},}$$

   consisting of one shorted put, one long call, and short one share of stock (to cover the shorted put). The net proceeds are $-5 + 3 + 50 = 48$, which are invested at 8%.

   B) Use your answer to (A) to compute the guaranteed profit after 6-months from your strategy.

   Solution: At expiration the value of the portfolio is

   $$\begin{aligned} V_\Pi &= \underbrace{-(48 - S_{1/2})^+}_{\text{short put}} + \underbrace{(S_{1/2} - 48)^+}_{\text{long call}} + \underbrace{(-S_{1/2})}_{\text{shorted stock}} + \underbrace{48e^{0.04}}_{\text{bank investment}}, \\ &= S_{1/2} - 48 - S_{1/2} + 48e^{0.04}, \\ &= 48(e^{0.04} - 1), \\ &\simeq \boxed{1.959.} \end{aligned}$$

4. You are given:

- The current price of the stock is 70.
- The stock pays continuous dividends at an annual rate of 8%.
- The continuously compounded risk-free interest rate is 4%.
- A 1-year American put option on the stock has a strike price of 69.

Determine the lowest possible price for this put option.

A) 0
B) 0.57
C) 1.02
D) 1.68
E) none of the above

Solution: The American put option must be worth at least as much as a European put option. By put-call parity for European options we know that

$$C^E - P^E = S_0 e^{-\delta \cdot T} - K e^{-r \cdot T} \quad \text{or} \quad P^E = C^E + K e^{-rT} - S_0 e^{-\delta T}$$

which, for $C^E \geq 0$, reduces to

$$P^E \geq K e^{-rT} - S_0 e^{-\delta T}.$$

Hence we have that

$$P^A \geq P^E \geq 69e^{-0.04} - 70e^{-0.08} \simeq \boxed{1.676.}$$

$\therefore$ **The correct answer is D.**

5. For two stocks X and Y with prices $X(t)$ and $Y(t)$, you know

- $X(0) = 40$.
- X pays dividends at a continuous annual rate of 2%.
- Y does not pay dividends.
- After 6 months, the possible prices for X and Y are

Outcome	Price of X	Price of Y
1	30	70
2	50	30

- The continuously compounded risk-free interest rate is 8%.

Determine the arbitrage-free initial price $Y(0)$ of stock Y.

Solution: Replicate one share of Y using a shares of X and b zero-coupon bonds. Then for outcomes 1 and 2 respectively we require

$$\underbrace{30e^{0.02/2}a + e^{0.08/2}b = 70}_{\text{Outcome 1}} \quad \text{and} \quad \underbrace{50e^{0.02/2}a + e^{0.08/2}b = 30}_{\text{Outcome 2}}.$$

Subtracting the first equation from the second yields

$$20e^{0.01}a = -40 \quad \Rightarrow \quad a = -2e^{-0.01} \simeq -1.98010.$$

Solving for b results in

$$b = e^{-0.04}\left(70 - 30e^{0.01}(-1.98010)\right) \simeq 124.90.$$

Therefore, the initial price of $Y(0)$ is

$$\boxed{Y(0) = -1.98010X(0) + 124.90(1) \simeq 45.70.}$$

6. You are given

   - The Black-Scholes framework holds.
   - The current exchange rate between euros and dollars is 0.85\$/€.
   - The annual volatility of the exchange rate is 10%.
   - The continuously compounded risk-free interest rate for dollars is 5%.
   - The continuously compounded risk-free interest rate for euros is 2%.

   Determine the premium for a dollar-denominated 1-year European call option on euros with a strike price of 0.9\$/€.

   A) 0.234
   B) 0.319
   C) 0.388
   D) 0.417
   E) 0.469

   Solution: The Garman-Kohlhagan formula states that the price of the call option within the BSM framework is

$$C = x_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$$

   where x_0 is the current exchange rage, K is the strike exchange rate, and

$$d_1 = \frac{\ln(x_0/K) + (r - r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \& \quad d_2 = d_1 - \sigma\sqrt{T}.$$

   Substituting the given information we find that

$$d_1 = \frac{\ln(0.85/0.9) + (0.05 - 0.02 + 0.1^2/2) \cdot 1}{0.1 \cdot \sqrt{1}} \simeq -0.2216 \quad \text{and} \quad d_2 \simeq -0.3216.$$

   Further, since $N(d_1) \simeq 0.4123$ and $N(d_2) \simeq 0.3739$ it follows that

$$C = 0.85 \cdot e^{-0.02} \cdot (0.4123) - 0.9 \cdot e^{-0.05} \cdot (0.3739) \simeq \boxed{0.234.}$$

   ∴ **The correct answer is A.**

7. A market-maker sells a 1-year European at-the-money put option on a non-dividend paying stock and Δ-hedges it. You are given

- The current price of the stock is 50.
- The continuously compounded risk-free interest rate is 8%.
- The volatility of the stock is 22%.
- The Black-Scholes framework holds.
- The option premium is 2.5632 and the option's Δ is -0.3179.

After 1 week the stock's price is 49. Determine the amount of money required to purchase additional shares of the stock to maintain the Δ-hedge after 1 week.

<u>*Solution:*</u> Δ for a put option is

$$\Delta(0) = -e^{-\delta} N(-d_1^{(0)}) = -0.3192 = -N(-d_1^{(0)}).$$

After 7 days,

$$d_1^{(7/365)} = \frac{\ln\left(\frac{49}{50}\right) + (0.08 - 0 + 0.22^2/2)(358/365)}{0.22\sqrt{358/365}} \simeq 0.37635.$$

Therefore, after 7 days, Δ for the shorted put is

$$\Delta(7/365) = -N(-d_1^{(7/365)}) = -N(-0.37635) \simeq -N(-0.38) = -0.3520.$$

Since the original hedge consisted of -0.3179 shares and the new hedge is -0.3520 shares it follows that an additional (-0.3520-(-0.3179)) = -0.0341 additional shares which costs

$$\boxed{(-0.0341)(49) = -1.6709.}$$

8. For European call and put options on a stock having strike price K and expiration time t:

(i) The stock follows the Black-Scholes framework with time-t price $S(t)$.
(ii) $S(0)e^{-\delta t} = 500$ and $Ke^{-rt} = 450.43$.
(iii) $N(d_1) = 0.8238$.
(iv) $\text{Var}\left[\ln(S(t)/S(0))\right] < 1$.

Calculate the price at time 0 of a portfolio consisting of two long calls and one shorted put.

<u>*Solution:*</u> Because $N(d_1) = 0.8238$ we know

$$d_1 = 0.93 = \frac{\ln\left(\frac{S}{K}\right) + (r - \delta + \sigma^2/2)t}{\sigma\sqrt{t}} = \frac{\ln\left(\frac{Se^{-\delta t}}{Ke^{-rt}}\right) + \sigma^2 t/2}{\sigma\sqrt{t}}$$

so that if $x = \sigma\sqrt{t}$ then

$$d_1 x = \ln(500/450.43) + x^2/2 \quad \text{or} \quad x^2 - 2(0.93)x + 2\ln(500/450.43) = 0.$$

Solving for x results in

$$x = \frac{1.86 \pm 1.62}{2} = 0.12 \quad \text{or} \quad 1.74.$$

Now, using the fact that the stock price follows geometric Brownian motion with

$$S(t) = S(0)e^{(r-\delta+\sigma^2/2)t+\sigma W(t)} \quad \Rightarrow \quad \ln(S(t)) = \ln(S(0)) + (r-\delta+\sigma^2/2)t + \sigma W(t)$$

so that

$$\text{Var}[\ln S(t)|S(0)] = \text{Var}[\ln(S(0)) + (r-\delta+\sigma^2/2)t + \sigma W(t)] = \sigma^2 t = x^2 < 1,$$

we conclude that only $x = \sigma\sqrt{t} = 0.12$ is possible. All that remains is to use Black-Scholes to compute the price of the associated puts and calls via

$$C = S(0)e^{-\delta t}N(d_1) - Ke^{-rt}N(d_2) = 500(0.8238) - 450.43N(0.93 - 0.12) = 55.6$$

and

$$P = Ke^{-rt}N(-d_2) - S(0)e^{-\delta t}N(-d_1) = 6.04.$$

Therefore, the price of two long calls and one short put is $\boxed{2(55.6) - 6.04 = 105.16.}$

9. Consider a 1-year, 2-period, binomial tree model of stock prices S_t where $S_0 = 50$, $d = 3/5$, and $u = 7/5$.

A) Draw the 1-year price tree for the stock.

Solution: The node values at $T = 1/2$ are

$$S_0 u = 70 \quad \text{and} \quad S_0 d = 30.$$

The node values at $T = 1$ are

$$S_2^{--} = S_0 d^2 = 18, \quad S_2^{+-} = S_2^{-+} = S_0 ud = 42, \quad \text{and} \quad S_2^{++} = S_0 u^2 = 98.$$

B) Given a risk-free annual continuous interest rate of 8% and a continuous dividend rate of 2%, use your answer to part (A) to find the arbitrage-free price of a 1-year floating strike, arithmetically averaged, European Asian put option.

Solution: The payoff for the floating strike Asian put is

$$\Lambda = (A_2 - S_2)^+ .$$

The payoff on the tree at $T = 1$ is therefore

PATH	S_1	S_2	$A_2 = (S_1 + S_2)/2$	$\Lambda = (A_2 - S_2)^+$
UU	70	98	84	0
UD	70	42	56	14
DU	30	42	36	0
UD	30	18	24	6

The risk-neutral probability is

$$\tilde{p} = \frac{e^{(r-\delta)\cdot(1/2)} - d}{u - d} = \frac{e^{(0.08-0.02)/2} - 0.6}{1.4 - 0.6} \simeq 0.5381.$$

It follows that the values of the put option at $T = 1/2$ are

$$P^E(\mathsf{U}) = e^{-0.08/2} \cdot (1 - 0.5381) \cdot 14 \simeq 6.213 \quad \text{and} \quad P^E = e^{-0.08/2} \cdot (1 - 0.5381) \cdot 6 \simeq 2.663.$$

Finally, the price of the put is

$$P^E = e^{-0.04} \left((0.5381) \cdot (6.213) + (1 - 0.5381) \cdot (2.663) \right) \simeq \boxed{4.394.}$$

10. A portfolio has a long European call option and a short European put option on the same stock, both with a strike price of 40 and one year to expiration. You are given:

- The stock's price is 45.
- The stock pays no dividends.
- The Black-Scholes framework holds.
- The continuously compounded risk-free interest rate is 4%.

Calculate the elasticity of the portfolio.

<u>*Solution:*</u> Let Π denote the portfolio. From put-call parity we know that the value of the portfolio has the general form:

$$V_{\Pi} = C - P = S - Ke^{-rT} = 45 - 40e^{-0.04} \simeq 6.5684.$$

Differentiating in S we find

$$\Delta_{\Pi} = \Delta_C - \Delta_P = 1.$$

It follows that the elasticity of the portfolio Ω_{Π} is

$$\Omega_{\Pi} = \frac{S \cdot \Delta_{\Pi}}{V_{\Pi}} = \frac{45 \cdot 1}{6.5684} \simeq \boxed{6.85096.}$$

11. The price of a non-dividend paying stock at time t, denoted by $S(t)$ with $S(0) = 40$, follows geometric Brownian motion

$$S(t) = S(0)e^{\left(r-\sigma^2/2\right)t+\sigma W(t)},$$

where $W(t)$ is a standard Brownian motion. A claim G on the stock is defined by

$$G = \left(S(1)S(2)S(3)\right)^{\frac{1}{3}}.$$

Assuming a continuous risk-free rate of 8% and a stock volatility of 30%, find the expected value of G.

<u>*Solution:*</u> The equation is geometric Brownian motion so that you know that

$$S(t) = S(0)e^{(0.08-0.3^2/2)t+0.3W(t)} = 40e^{0.035t+0.3W(t)}$$

and

$$
\begin{aligned}
G &= (S(1)S(2)S(3))^{\frac{1}{3}}, \\
&= \left(S^3(0)e^{0.035(1+2+3)+0.3(W(1)+W(2)+W(3))}\right)^{\frac{1}{3}}, \\
&= S(0)e^{0.07+0.1(W(1)+W(2)+W(3))}.
\end{aligned}
$$

Because $W(0) = 0$ we have

$$
\begin{aligned}
X = W(1) + W(2) + W(3) &= W(1) - W(0) + (W(2) - W(1) + W(1) - W(0)) \\
&\quad + (W(3) - W(2) + W(2) - W(1) + W(1) - W(0)), \\
&= 3(W(1) - W(0)) + 2(W(2) - W(1)) + (W(3) - W(2)),
\end{aligned}
$$

which is the sum of three normal and independent random variables. In particular, the mean of the previous expression is 0 and the variance is $3^2+2^2+1^2 = 14$. Therefore, G is lognormal with parameters $\mu = 0.07$ and $\sigma^2 = 0.1^2(14) = 0.14$. Finally, since the expected value of a lognormal random variable e^X is $e^{\mu+\sigma^2/2}$ we have that

$$
\boxed{E[G] = S(0)e^{\mu+\sigma^2/2} = 40e^{0.07+0.14/2} = 40e^{0.14} \simeq 46.01.}
$$

12. It is known that for 1-year European put options, the arbitrage-free price $P^E(K)$ satisfies

$$
P^E(40) = 4 \quad \text{and} \quad P^E(50) = 9.
$$

A) Use convexity to determine an upper bound on the price of a European put option with strike price 44.

<u>*Solution:*</u> Letting $K_1 = 40$, $K_2 = 44$, and $K_3 = 50$, solving

$$
\lambda K_1 + (1-\lambda)K_3 = 40\lambda + (1-\lambda)50 = 50 - 10\lambda = 44
$$

so that $\lambda = 3/5$ and $1 - \lambda = 2/5$. Therefore

$$
P^E(44) = P^E\left((3/5)40 + (2/5)(50)\right) \le (3/5)P^E(40) + (2/5)P^E(50) = (3/5)4 + (2/5)9
$$

so that

$$
\boxed{P^E(44) \le 6.}
$$

B) What is a lower bound to $P^E(44)$? Be sure to fully support your answer.

<u>*Solution:*</u> Immediately we know from the fact that $P^E(K)$ is an increasing function of K that $P^E(44) \ge P^E(40) = 4$. Recall from monotonicity that $K_1 < K_2$ also requires that

$$
0 \le P^E(K_2) - P^E(K_1) \le K_2 - K_1.
$$

Therefore, choosing $K_2 = 50$ and $K_2 = 44$ we get

$$P^E(50) - P^E(44) \leq 50 - 44 = 6 \quad \Rightarrow \quad P^E(50) - 6 \leq P^E(44)$$

or

$$P^E(44) \geq 9 - 6 = 3.$$

We conclude that the best inequality given the information presented is

$$\boxed{4 \leq P^E(44) \leq 6.}$$

13. GZHU stock is currently worth \$60 a share. The continuously compounded risk-free rate is $r = 6\%$, the annual continuously compounded dividend yield is $\delta = 2\%$, and the annualized standard deviation of the continuously compounded stock return is $\sigma = 15\%$.

   A) Using the Cox-Ross-Rubenstein model, find up and down factors u and d for a 2-period binomial model over the course of one year.

   <u>*Solution:*</u> Because $h = 1/2$ we have

$$u = e^{\sigma\sqrt{h}} = e^{(0.15)/\sqrt{2}} \simeq 1.111895 \quad \text{and} \quad d = e^{-\sigma\sqrt{h}} = \frac{1}{u} \simeq 0.899365$$

   so that

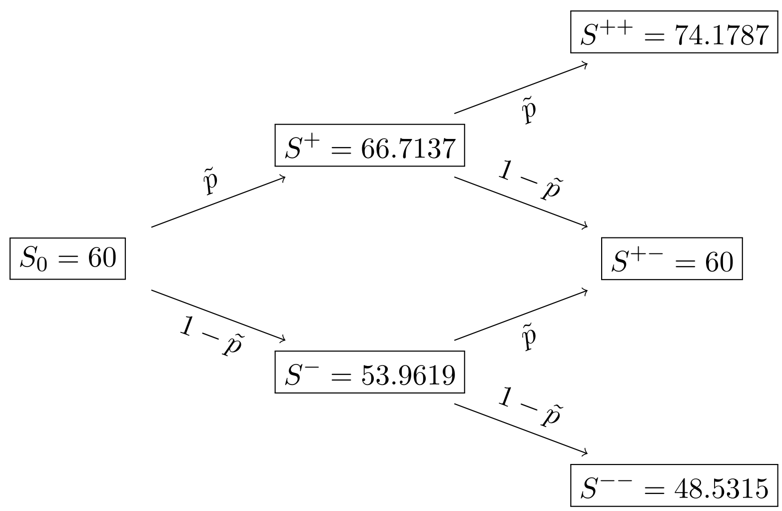
$$\boxed{u \simeq 1.111895 \quad \text{and} \quad d \simeq 0.899365.}$$

B) Use your answer to part (A) to draw the 2-period stock price tree.

Solution: Using

$$S_{t+1}^{-} = S_t d \quad \text{and} \quad S_{t+1}^{+} = S_t u$$

for $t = 0$ and $t = 1$ it follows that



C) Use your answers to parts (A) & (B) to find the arbitrage free price of a 1-year floating strike lookback call option.

Solution: At expiration, the value and payoff of the European floating strike lookback call option is $\Lambda(S_T) = (S_T - s_1)^+$, where $s_1 = \min_{0 \le t \le 1} S_t$. Using the price tree from part (B) we have the following path dependent payoffs at $T = 1$:

Path	S_0	$S_{1/2}$	S_1	m_1	$\Lambda = (S_1 - m_1)^+$
UU	60	66.7137	74.1787	60	14.1787
UD	60	66.7137	60	60	0
DU	60	53.9619	60	53.9619	6.0381
DD	60	53.9619	48.5315	48.5315	0

The risk-neutral probability for this binomial model is

$$\tilde{p} = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(.06-.02)/2} - 0.899365}{1.111895 - 0.899365} \simeq 0.56856.$$

Therefore, letting $h = 1/2$ we have

$$V^E\left(S_{1/2}^{+} = 66.7137\right) = e^{-rh}\tilde{E}[S_1] = e^{-rh}(\tilde{p} \cdot (14.1787) + (1 - \tilde{p}) \cdot 0) \simeq 7.8232$$

and

$$V^E\left(S_{1/2}^{-}=53.9619\right)=e^{-rh}\tilde{E}[S_1]=e^{-rh}(\tilde{p}\cdot(6.0381)+(1-\tilde{p})\cdot 0)\simeq 3.3316.$$

Finally, the arbitrage-free price of the derivative at time $t=0$ is

$$\boxed{V^E(S_0)=e^{-rh}\left(\tilde{p}\,(7.8232)+(1-\tilde{p})\,(3.3316)\right)\simeq 5.7114.}$$

14. You are considering the purchase of 100 units of a 3-month 25-strike European put option on a stock. You know that

- The Black-Scholes framework holds.
- The stock currently sells for \$20.
- The stock's volatility is 24%.
- The stock pays dividends continuously with a dividend yield of 3%.
- The continuously compounded risk-free interest rate is 5%.

Find the price of a block of 100 of these put options. Be sure to show **ALL** of your work.

A) 336.2
B) 354.0
C) 381.9
D) 442.4
E) 487.4

<u>*Solution:*</u> Given

$$S_0=\$20,\quad T=\frac{1}{4},\quad K=\$25,\quad r=0.05,\quad \delta=0.03,\quad \text{and}\quad \sigma=0.24$$

it follows that

$$d_1=\frac{\ln\left(\frac{20}{25}\right)+\left(0.05-0.03+\frac{(0.24)^2}{2}\right)\left(\frac{1}{4}\right)}{0.24\sqrt{\frac{1}{4}}}\simeq -1.75786\quad\text{and}\quad d_2=d_1-\frac{0.24}{\sqrt{4}}\simeq -1.87786.$$

Therefore, rounding $d_1\doteq -1.76$ and $d_2\doteq -1.88$ the cumulative normal distribution function table gives

$$N(-d_1)=N(1.76)=\int_{-\infty}^{1.76}\frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\,d\xi=0.9606\quad\text{and}\quad N(-d_2)=N(1.88)=0.9699.$$

Hence, the Black-Scholes formula for the price of a put yields

$$\begin{aligned} P^E &= Ke^{-rT}N(-d_2) - S_0e^{-\delta T}N(-d_1), \\ &= 25e^{-.05/4}(0.9699) - 20e^{-0.03/4}(0.9606), \\ &= 4.87387. \end{aligned}$$

Therefore, the arbitrage free price of 100 puts is

$$\boxed{100P^E = \$487.387.}$$

The correct answer is E.

15. Consider a 3-month \$41.50-strike American call option on a non-dividend paying stock. Suppose

- the Black-Scholes framework holds;
- the stock currently sells for \$40;
- the stock's volatility is 30%;
- the current value of Δ_{call} is 1/2;

A) Find the continuously compounded risk-free interest rate r.

Solution: For a non-dividend paying stock, $\delta = 0$ so that for $\sigma = 0.3$, $K = 41.5$, and $S_0 = 40$ we have

$$\Delta_{\text{call}} = e^{-\delta(T-t)}N(d_1) = e^0 N(d_1) = \int_{-\infty}^{d_1} \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\, d\xi = \frac{1}{2} \quad \Longrightarrow \quad d_1 = 0.$$

Therefore

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = 0.$$

Multiplying through by $\sigma\sqrt{T-t}$ and solving for r yields

$$r = -\frac{\ln\left(\frac{S_t}{K}\right)}{T-t} - \sigma^2/2 = -\frac{\ln\left(\frac{40}{41.5}\right)}{\frac{1}{4} - 0} - (.3)^2/2 \simeq 0.1022559$$

so that

$$\boxed{r \simeq 10.22559\%.}$$

B) Use your answer to part (A) to find the current price of the call option. Be sure to explain your reasoning.

Solution: First note that since this is a non-dividend paying stock, early exercise is never advantageous so

$$C^E = C^A.$$

Consequently, we can use Black-Scholes' European option pricing formula to find C^E, and then conclude by setting it equal to C^A. Now, because $d_1 = 0$ it follows that

$$d_2 = d_1 - \sigma\sqrt{\frac{1}{4}} = -\frac{0.3}{2} = -0.15.$$

Therefore, from the cumulative normal distribution function tables we have

$$N(d_1) = N(0) = \Delta_{\text{call}} = 0.5 \quad \text{and} \quad N(d_2) = N(-0.15) = 0.4404.$$

Using part (A),

$$t = 0, \quad T = \frac{1}{4}, \quad \text{and} \quad \sigma = 0.3,$$

the Black-Scholes formula gives

$$\begin{aligned} C^E &= S_0 e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2) \\ &= 40e^0 N(0) - 41.5e^{-.1022559/4} N(-0.15) \\ &= \frac{40}{2} - 41.5e^{-0.1022559/4}(0.4404) \\ &\simeq 2.1847. \end{aligned}$$

Therefore, the final answer is

$$\boxed{C^A = C^E \simeq \$2.1847.}$$

16. You are given the following information for two European options on a stock priced using the Black-Scholes formula:

	45-strike put	50-strike call
Price	0.0211	10.2270
Δ	-0.0082	0.9300
Γ	0.0030	0.0160

The stock price is 60. Determine the number of shares of stock and the number of 50-strike calls one must buy or sell to Δ-Γ hedge a sale of a 45-strike put.

Solution: The portfolio Π is

$$\Pi = \{-1, x, y\}$$

where x is the number of shares of stock to buy/sell and y is the number of 50-strike calls to buy/sell. Δ-hedging the portfolio requires that

$$\Delta_\Pi = -\frac{\partial P^E(45)}{\partial S} + x\left(\frac{\partial S}{\partial S}\right) + y\left(\frac{\partial C^E(50)}{\partial S}\right) = -\Delta_{45} + x + y\Delta_{50} = 0.$$

Further, Γ-hedging the portfolio requires that

$$\Gamma_\Pi = -\frac{\partial^2 P^E(45)}{\partial S} + x\left(\frac{\partial^2 S}{\partial S^2}\right) + y\left(\frac{\partial^2 C^E(50)}{\partial S^2}\right) = -\Gamma_{45} + 0 + y\Gamma_{50} = 0.$$

Therefore,

$$y = \frac{\Gamma_{45}}{\Gamma_{50}} = \frac{0.0030}{0.0160} = 0.1875$$

and

$$x = -0.0082 - y(0.9300) = -0.0082 - (0.1875)(0.9300) = -0.182575.$$

Hence you should

Short 0.182575 shares of the underlying stock and buy 0.1875 of the 50-strike calls.

17. Consider a European call option on a stock following the Black-Scholes framework. The option expires in one year. Using the Black-Scholes formula for the option, you obtain

$$N(d_1) = 0.6591 \quad \text{and} \quad N(d_2) = 0.3409.$$

A) Find the volatility σ of the stock.

Solution: Using the binomial table it follows from $T = 1$ that

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + r - \delta + \sigma^2/2}{\sigma} = 0.41$$

and

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + r - \delta - \sigma^2/2}{\sigma} = -0.41$$

Since

$$d_2 = -0.41 = d_1 - \sigma\sqrt{T} = d_1 - \sigma = 0.41 - \sigma$$

it follows that

$$\boxed{\sigma = 0.82.}$$

B) Use the concept of call elasticity Ω_{Call} to deduce the volatility of the option.

Solution: Recall that

$$\sigma_{\text{call}} = \sigma_{\text{stock}}|\Omega| \quad \text{where} \quad \Omega = \frac{S\Delta}{C}.$$

Using the Black-Scholes formula, the equation for Ω becomes

$$\Omega = \frac{S\Delta}{Se^{-\delta h}N(d_1) - Ke^{-rh}N(d_2)} = \frac{Se^{-\delta h}N(d_1)}{Se^{-\delta h}N(d_1) - Ke^{-rh}N(d_2)}.$$

From part (A) we have

$$\frac{\ln\left(\frac{S}{K}\right) + r - \delta + \sigma^2/2}{\sigma} = 0.41 = -\frac{\ln\left(\frac{S}{K}\right) + r - \delta - \sigma^2/2}{\sigma}.$$

Canceling $\sigma > 0$ we get

$$\ln\left(\frac{S}{K}\right) + r - \delta + \sigma^2/2 = -\left(\ln\left(\frac{S}{K}\right) + r - \delta - \sigma^2/2\right) = -\ln\left(\frac{S}{K}\right) - r + \delta + \sigma^2/2.$$

Canceling $\sigma^2/2$ it follows that

$$\ln\left(\frac{S}{K}\right) + r - \delta = 0 \quad \Rightarrow \quad \frac{S}{K} = e^{\delta - r} \quad \text{or} \quad Se^{-\delta} = Ke^{-r}.$$

Hence

$$
\begin{aligned}
\Omega &= \frac{Se^{-\delta h}N(d_1)}{Se^{-\delta h}N(d_1) - Ke^{-rh}N(d_2)}, \\
&= \frac{Se^{-\delta}N(d_1)}{Se^{-\delta h}N(d_1) - Se^{-\delta h}N(d_2)}, \\
&= \frac{N(d_1)}{N(d_1) - N(d_2)}, \\
&= \frac{0.6591}{0.6591 - 0.3409}, \\
&\simeq 2.071339.
\end{aligned}
$$

Therefore,

$$
\sigma_{\text{call}} = |\Omega_{\text{call}}|\,\sigma_{\text{stock}} = (2.071339)(0.82) \simeq 1.69849798.
$$

18. Let S_t be the price of a stock at time t. S_t follows a lognormal model and you are given the following information:

    (i) The continuously compounded expected annual rate of return on the stock is 15%.

    (ii) The continuously compounded dividend rate is 2%.

    (iii) The stock's volatility is 30%.

    (iv) $S_0 = 80$.

    Calculate the probability that a 4-year European call with strike $K = 130$ has a payoff of 20 at expiration.

    A) 0.227

    B) 0.316

    C) 0.392

    D) 0.425

    E) 0.504

    Solution: For a 4-year European call with strike $K = 130$ to have a payoff of at least 20 at expiration, we require that

    $$
    (S_4 - 130)^+ \geq 20 \quad \text{or} \quad S_4 \geq 150.
    $$

    Therefore, using the parameters in the problem, we need

$$
\begin{aligned}
\mathbb{P}(S_4 \geq 150) &= \mathbb{P}\left(\ln(S_4) \geq \ln(150)\right), \\
&= \mathbb{P}\left(\ln(S_4) - \ln(80) \geq \ln(150) - \ln(80)\right), \\
&= \mathbb{P}\left(\ln\left(\frac{S_4}{80}\right) \geq \ln\left(\frac{150}{80}\right)\right), \\
&= \mathbb{P}\left(\ln\left(\frac{S_4}{80}\right) - (\alpha - \delta - \sigma^2/2)t \geq \ln\left(\frac{150}{80}\right) - (\alpha - \delta - \sigma^2/2)t\right), \\
&= \mathbb{P}\left(\frac{\ln\left(\frac{S_4}{80}\right) - (\alpha - \delta - \sigma^2/2)t}{\sigma\sqrt{t}} \geq \frac{\ln\left(\frac{150}{80}\right) - (\alpha - \delta - \sigma^2/2)t}{\sigma\sqrt{t}}\right), \\
&= \mathbb{P}\left(Z \geq \frac{\ln\left(\frac{150}{80}\right) - (\alpha - \delta - \sigma^2/2)t}{\sigma\sqrt{t}}\right), \\
&= \mathbb{P}\left(Z \geq \frac{\ln\left(\frac{150}{80}\right) - (0.15 - 0.02 - 0.3^2/2)(4)}{0.3\sqrt{4}}\right), \\
&= \mathbb{P}(Z \geq 0.4810), \\
&\simeq 1 - \mathbb{P}(Z < 0.48), \\
&= 1 - 0.6844, \\
&= \boxed{0.3156.}
\end{aligned}
$$

The correct answer is B.

19. Consider a European call option and a European put option on a non-dividend paying stock. The current stock price is \$60, the call option currently sells for \$0.15 more than the put option. Both the call and put option have strike price $K = \$70$ and will expire in 4 years. Find the continuously compounded annual risk-free interest rate r.

A) 3.92%

B) 4.22%

C) 4.79%

D) 5.01%

E) 5.66%

Solution: Put-Call Parity (PCP) states for a non-dividend paying stock that

$$C^E(K) - P^E(K) = PV\left[F_{0,T} - K\right] = e^{-rT}\left(S_0 e^{rT} - K\right) = S_0 - Ke^{-rT}.$$

Since $S_0 = 60$, $T = 4$, and

$$C^E(70) - P^E(70) = 0.15,$$

it follows that we just need to solve

$$0.15 = 60 - 70e^{-4r}$$

for r. Clearly,

$$70e^{-4r} = 60 - 0.15 = 59.85 \quad \Rightarrow \quad r = -\frac{1}{4}\ln\left(\frac{59.85}{70}\right) \simeq 0.03916345\ldots$$

or

$$\boxed{r \simeq 3.916\%}$$

The correct answer is A.

20. You are given the following information for a stock with price $S(t)$ at time t that follows the Black-Scholes framework:

  (i) The time t price of a derivative security of the stock is $S(t)^a e^{0.0375t}$ with $a < 0$;
  (ii) The continuously compounded risk-free interest rate is 0.05;
  (iii) Neither the stock nor the derivative security pays dividends;
  (iv) The volatility of the stock is 40%.

Use the Black-Scholes partial differential equation to determine a.

<u>*Solution:*</u> Let $V(S,t) = S^a e^{\alpha t}$ for $\alpha = 0.0375$. Then

$$V_t = \alpha S^a e^{\alpha t}, \quad V_S = aS^{a-1}e^{\alpha t}, \quad \text{and} \quad V_{SS} = a(a-1)S^{a-2}e^{\alpha t}.$$

The Black-Scholes PDE then gives

$$\begin{aligned} rV &= rS^a(t)e^{\alpha t}, \\ &= \alpha S^a e^{\alpha t} + rS(t)\left(aS^{a-1}e^{\alpha t}\right) + \frac{1}{2}\sigma^2\left(a(a-1)S^{a-2}(t)e^{\alpha t}\right), \\ &= S^a(t)e^{\alpha t}\left(\alpha + ar + a\sigma^2(a-1)/2\right), \\ &= S^a(t)e^{\alpha t}\left(\alpha + \sigma^2 a^2/2 + a(r - \sigma^2/2)\right). \end{aligned}$$

Assuming that $S(t) > 0$, cancelling it and $e^{\alpha t}$ we require

$$\sigma^2 a^2 + a\left(2r - \sigma^2\right) + 2(\alpha - r) = 0.$$

Solving we get for $\sigma = 0.4$, $r = 0.05$, and $\alpha = 0.0375$ that

$$a = \frac{(\sigma^2 - 2r) \pm \sqrt{(2r - \sigma^2)^2 - 8\sigma^2(\alpha - r)}}{2\sigma^2} = -0.25,\ 0.625.$$

Since $a < 0$ it follows that $\boxed{a = -0.25.}$

21. $X(t)$ follows the process

$$dX(t) = 0.5\,t\,dt + 0.2\,dW(t),$$

where $W(t)$ is standard Brownian motion. Another process $Y(t) = t\,e^{X(t)}$ satisfies

$$dY(t) = \alpha(t)\,Y(t)\,dt + \sigma(t)\,Y(t)\,dW(t).$$

Determine the lowest value of t for which $\alpha(t) = 2$.

A) 0.59

B) 0.65

C) 0.71

D) 0.77

E) 0.85

Solution: Let $f(X,\,t) = Y(t) = te^X$ so that $f_t = e^X = Y/t$, $f_X = te^X = Y$, and $f_{XX} = te^X = Y$. SInce $(dX)^2 = (0.2)^2(dW)^2 = 0.04\,dt$, it follows from Itô's Lemma that

$$\begin{aligned} df &= f_X\,dX + f_t\,dt + (f_{XX}/2)\,(dX)^2 = Y\,dX + \frac{Y}{t}\,dt + \frac{Y}{2}\cdot(dX)^2 = Y\,dX + \left(\frac{Y}{t} + \frac{Y}{2}\cdot 0.04\right)\,dt, \\ &= Y\,(0.5\,t\,dt + 0.2\,dW) + \left(\frac{Y}{t} + 0.02\,Y\right)\,dt = Y\left(0.5\,t + \frac{1}{t} + 0.02\right)\,dt + 0.2\,Y\,dW. \end{aligned}$$

It follows that $\alpha(t) = 0.5t + t^{-1} + 0.02$. Setting $\alpha(t) = 2$ yields

$$2 = 0.5\,t + \frac{1}{t} + 0.02 \quad\Rightarrow\quad 0.5t^2 - 1.98\,t + 1 = 0 \quad\Rightarrow\quad t = \frac{1.98 \pm \sqrt{1.98^2 - 4(0.5)(1)}}{2(0.5)},$$

or $t = 0.594215,\ 3.36578$. We conclude that $\boxed{t \simeq 0.59.}$

$\therefore$ **the correct answer is A.**

22. For two European call options on the same stock, C_1 and C_2:

- The stock price is 45.
- The price of C_1 is 10.86.
- The delta of C_1 is 0.6252.
- Mason buys one call of each type. Mason's portfolio has elasticity 2.90.
- Shay buys one C_1 and two C_2's. Shay's portfolio has elasticity 3.03.
- Sarah buys one C_1 and sells one C_2.

Determine the elasticity of Sarah's portfolio.

Solution: Recall that, in general, given portfolio Π, $\Omega_\Pi = S \cdot \Delta_\Pi / V_\Pi$, where S is the value of the underlying asset and V_Π is the value of the portfolio. Hence, for Mason we have

$$\Omega_M = \frac{S\,(\Delta_1 + \Delta_2)}{C_1 + C_2} = \frac{45\,(0.6252 + \Delta_2)}{10.86 + C_2} = 2.90,$$

while, for Shay we observe

$$\Omega_{Sh} = \frac{S\,(\Delta_1 + 2\,\Delta_2)}{C_1 + 2C_2} = \frac{45\,(0.6252 + 2\,\Delta_2)}{10.86 + 2C_2} = 3.03.$$

Solving the above two equations for C_2 and Δ_2 yields $C_2 = 7.49308$ and $\Delta_2 = 0.557554$. Therefore, Sarah's portfolio has elasticity

$$\Omega_{Sa} = \frac{S\,(\Delta_1 - \Delta_2)}{C_1 - C_2} = \frac{45\,(0.6252 - 0.557554)}{10.86 - 7.49308} \simeq \boxed{0.904111.}.$$

23. The time-t price of a stock is $S(t)$. You are given

- $S(t)$ follows a geometric Brownian motion.
- $S(0) = 1.2$.
- $\mathbb{P}(S(1) > 1.2) = 0.60642$.
- $\mathbb{P}(S(2) > 1.44) = 0.34827$.

Determine $\sigma^2_{S(1)}$.

A) 0.03
B) 0.04
C) 0.05
D) 0.06
E) 0.07

<u>*Solution:*</u> We are given that $S(t)$ follows $dS = \mu\, S\, dt + \sigma\, S\, dW$, which has solution $S(t) = S(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)}$. Since

$$\begin{aligned}
\mathbb{P}(S(1) > 1.2) &= \mathbb{P}\left(1.2e^{(\mu-\sigma^2/2)(1)+\sigma\sqrt{1}\,Z} > 1.2\right) = \mathbb{P}\left(e^{(\mu-\sigma^2/2)+\sigma\, Z} > 1\right), \\
&= \mathbb{P}\left((\mu-\sigma^2/2)+\sigma\, Z > 0\right) = \mathbb{P}\left(Z > \frac{(\sigma^2/2)-\mu}{\sigma}\right), \\
&= 1 - \mathbb{P}\left(Z \le \frac{(\sigma^2/2)-\mu}{\sigma}\right) = 0.60642,
\end{aligned}$$

so that

$$\mathbb{P}\left(Z \le \frac{(\sigma^2/2)-\mu}{\sigma}\right) = 0.39358 \quad \Rightarrow \quad \frac{\sigma^2/2-\mu}{\sigma} = -0.27 \quad \Rightarrow \quad \boxed{\sigma^2 + 0.54\,\sigma - 2\mu = 0.}$$

Similarly,

$$\begin{aligned}
\mathbb{P}(S(2) > 1.44) &= \mathbb{P}\left(1.2e^{(\mu-\sigma^2/2)(2)+\sigma\sqrt{2}\,Z} > 1.44\right) = \mathbb{P}\left(e^{(\mu-\sigma^2/2)(2)+\sigma\sqrt{2}\,Z} > 1.2\right), \\
&= \mathbb{P}\left((\mu-\sigma^2/2)(2)+\sigma\sqrt{2}\,Z > \ln 1.2\right) = \mathbb{P}\left(Z > \frac{\sigma^2-2\mu+\ln 1.2}{\sigma\sqrt{2}}\right), \\
&= 1 - \mathbb{P}\left(Z \le \frac{\sigma^2-2\mu+\ln 1.2}{\sigma\sqrt{2}}\right) = 0.34827,
\end{aligned}$$

so that

$$\mathbb{P}\left(Z \le \frac{\sigma^2 - 2\mu + \ln 1.2}{\sigma\sqrt{2}}\right) = 0.65173 \quad \Rightarrow \quad \frac{\sigma^2 - 2\mu + \ln 1.2}{\sigma\sqrt{2}} = 0.39,$$

or

$$\boxed{\sigma^2 \;-\; 0.39\sqrt{2}\,\sigma - 2\mu \;+\; \ln 1.2 = 0.}$$

Solving the two boxed equations for μ and σ yields $(\mu,\, \sigma) \simeq (0.059048,\, 0.167031)$. Finally, recall that the variance of a lognormal random variable of the form $Y = e^X$, where $X \sim N(\mu,\, \sigma^2)$, is $\sigma_Y^2 = \left(e^{\sigma^2} - 1\right) e^{2\mu+\sigma^2}$. Since $S(1)/S(0) = e^X$ with

$$X \simeq N\left(0.059048 - (0.167031)^2/2,\, (0.167031)^2\right) = N\left(0.0450983,\, 0.167031^2\right),$$

we conclude that

$$\sigma_{S(1)}^2 = S(0)^2\left(e^{\sigma^2} - 1\right) e^{2\mu+\sigma^2} = (1.2)^2\left(e^{0.167031^2} - 1\right) e^{2\cdot 0.0450983+0.167031^2} \simeq \boxed{0.0458477.}$$

$\therefore$ **the correct answer is C.**