Math 458 - Financial Math for Actuaries II - Practice Final Exam

1. total Please circle either T (true) or F (false) for each of the below statements. Each correct answer is worth 1 point. There is no partial credit or penalty for guessing.

I) T F A short European put option with strike $K_1 = 20$ combined with a long European put option with strike $K_2 = 30$ forms a long bull spread.

II) T F If the continuous risk-free rate is always equal to the continuous dividend rate on a stock with price S_t at time t, then arbitrage opportunities do not exist for forward contracts.

III) T F It is impossible for an American call to have premium 20 at $t = 0$ when the initial stock price is 15.

IV) T F It is possible for $0 < d < u < 1$ in a risk-neutral binomial tree.

V) T F In a symmetric random walk M_k with $M_0 = 0$, it is possible for $M_{17} = 0$.

VI) T F It is possible, when $K = 4$ and $S(0) = 5$, for a European call option to have premium $C^E = 1$ while for the related up-and-in call, $C^{ui} = 0.32$.

VII) T F Within the BSM framework, θ for a long put option can be positive or negative.

VIII) T F The Δ-Γ-θ approximation can be used to approximate the European call premium difference $C(S_{t+h}) - C(S_t)$ if $S_{t+h} = S_t$ for $h > 0$. That is, the approximation is valid if the stock price does not change but time *does* change.

IX) T F Within the BSM framework, it is possible that $\Delta = 0$ for a European put.

X) T F On a Jarrow-Rudd binomial tree, geometrically averaged strike Asian European options are not path dependent.

2. total points - 1 point each) Please circle either T (true) or F (false) for each of the below statements. All prices are no-arbitrage prices.

I) T F Combining a short straddle with strike price $K_2 = \$60$ with a long strangle with strikes $K_1 = 30$ and $K_3 = 90$ forms a symmetric butterfly spread.

II) T F If $r = \delta$ for a stock with price S_t, then the forward price $F_{0,T}$ is equal to the asset price S_0.

III) T F It is never advantageous to exercise an American call option early.

IV) T F The arithmetic Brownian motion

$$X(t) = 5 + 13t + 2W(t)$$

satisfies

$$\mathrm{Var}(X(t+3) - X(t)) = 6.$$

V) T F In the Cox-Ross-Rubenstein binomial model, if $u = 1.2$ then $d = 0.8$.

VI) T F In a 5-period binomial tree with initial stock price $S_0 = \$100$, expiration $T = 1$ year, $u = 1.1$, and $d = 0.9$, the second highest value of S_1 is \$131.769.

VII) T F Given a risk-free rate $r > 0$ and continuous dividend rate $\delta > 0$, one way to compute the risk-neutral probability $\tilde{p}$ for the binomial model is to require that the risk-neutral expected value of the stock S_{t+1} is equal to the forward price of the stock. That is, solve

$$\tilde{E}[S_{t+1}] = \tilde{p}(uS_t) + (1 - \tilde{p})(dS_t) = S_t e^{(r-\delta)h}.$$

for $\tilde{p}$.

VIII) T F If the $C(S)$ is the price of a European call and you know that $C(40) = 2.7804$, $\Delta(40) = 0.5824$, and $\Gamma(40) = 0.0652$, then $C(40.75) \approx 3.2355$. Assume that all other problem parameters (including time) are held constant.

IX) T F Because the Black-Scholes formula gives the arbitrage-free price of a call option in continuous time, you cannot use it to price a call on a stock where only discrete dividends are paid (e.g., owning a stock pays dividends only twice/year).

X) T F The fact that the “Greek” $\Gamma > 0$ is always true guarantees that C^E and $-P^E$ are convex.

3. You are given the following information on a non-dividend paying stock:

- The stock price is 50.
- The price of a 6-month European call option with an exercise price of 48 is 5.
- The price of a 6-month European put option with an exercise price of 48 is 3.
- The risk-free interest rate is constant 8%, compounded continuously.

A) There is an arbitrage opportunity involving buying or selling one share of stock and buying or selling puts and calls. Construct an arbitrage portfolio Π for this scenario.

B) Use your answer to (A) to compute the guaranteed profit after 6-months from your strategy.

4. You are given:

- The current price of the stock is 70.
- The stock pays continuous dividends at an annual rate of 8%.
- The continuously compounded risk-free interest rate is 4%.
- A 1-year American put option on the stock has a strike price of 69.

Determine the lowest possible price for this put option.

A) 0
B) 0.57
C) 1.02
D) 1.68
E) none of the above

5. For two stocks X and Y with prices $X(t)$ and $Y(t)$, you know

- $X(0) = 40$.
- X pays dividends at a continuous annual rate of 2%.
- Y does not pay dividends.
- After 6 months, the possible prices for X and Y are

Outcome	Price of X	Price of Y
1	30	70
2	50	30

- The continuously compounded risk-free interest rate is 8%.

Determine the arbitrage-free initial price $Y(0)$ of stock Y.

6. You are given

- The Black-Scholes framework holds.
- The current exchange rate between euros and dollars is 0.85\$/€.
- The annual volatility of the exchange rate is 10%.
- The continuously compounded risk-free interest rate for dollars is 5%.
- The continuously compounded risk-free interest rate for euros is 2%.

Determine the premium for a dollar-denominated 1-year European call option on euros with a strike price of 0.9\$/€.

A) 0.234
B) 0.319
C) 0.388
D) 0.417
E) 0.469

7. A market-maker sells a 1-year European at-the-money put option on a non-dividend paying stock and Δ-hedges it. You are given

- The current price of the stock is 50.
- The continuously compounded risk-free interest rate is 8%.
- The volatility of the stock is 22%.
- The Black-Scholes framework holds.
- The option premium is 2.5632 and the option's Δ is -0.3179.

After 1 week the stock's price is 49. Determine the amount of money required to purchase additional shares of the stock to maintain the Δ-hedge after 1 week.

8. For European call and put options on a stock having strike price K and expiration time t:

(i) The stock follows the Black-Scholes framework with time-t price $S(t)$.
(ii) $S(0)e^{-\delta t} = 500$ and $Ke^{-rt} = 450.43$.
(iii) $N(d_1) = 0.8238$.
(iv) $\text{Var}\left[\ln(S(t)/S(0))\right] < 1$.

Calculate the price at time 0 of a portfolio consisting of two long calls and one shorted put.

9. Consider a 1-year, 2-period, binomial tree model of stock prices S_t where $S_0 = 50$, $d = 3/5$, and $u = 7/5$.

   A) Draw the 1-year price tree for the stock.

   B) Given a risk-free annual continuous interest rate of 8% and a continuous dividend rate of 2%, use your answer to part (A) to find the arbitrage-free price of a 1-year floating strike, arithmetically averaged, European Asian put option.

10. A portfolio has a long European call option and a short European put option on the same stock, both with a strike price of 40 and one year to expiration. You are given:

    - The stock's price is 45.
    - The stock pays no dividends.
    - The Black-Scholes framework holds.
    - The continuously compounded risk-free interest rate is 4%.

    Calculate the elasticity of the portfolio.

11. The price of a non-dividend paying stock at time t, denoted by $S(t)$ with $S(0) = 40$, follows geometric Brownian motion

$$S(t) = S(0)e^{\left(r-\sigma^2/2\right)t+\sigma W(t)},$$

    where $W(t)$ is a standard Brownian motion. A claim G on the stock is defined by

$$G = \left(S(1)S(2)S(3)\right)^{\frac{1}{3}}.$$

    Assuming a continuous risk-free rate of 8% and a stock volatility of 30%, find the expected value of G.

12. It is known that for 1-year European put options, the arbitrage-free price $P^E(K)$ satisfies

$$P^E(40) = 4 \quad \text{and} \quad P^E(50) = 9.$$

    A) Use convexity to determine an upper bound on the price of a European put option with strike price 44.

    B) What is a lower bound to $P^E(44)$? Be sure to fully support your answer.

13. GZHU stock is currently worth \$60 a share. The continuously compounded risk-free rate is $r = 6\%$, the annual continuously compounded dividend yield is $\delta = 2\%$, and the annualized standard deviation of the continuously compounded stock return is $\sigma = 15\%$.

    A) Using the Cox-Ross-Rubenstein model, find up and down factors u and d for a 2-period binomial model over the course of one year.

    B) Use your answer to part (A) to draw the 2-period stock price tree.

    C) Use your answers to parts (A) & (B) to find the arbitrage free price of a 1-year floating strike lookback call option.

14. You are considering the purchase of 100 units of a 3-month 25-strike European put option on a stock. You know that

    - The Black-Scholes framework holds.
    - The stock currently sells for \$20.
    - The stock's volatility is 24%.
    - The stock pays dividends continuously with a dividend yield of 3%.
    - The continuously compounded risk-free interest rate is 5%.

    Find the price of a block of 100 of these put options. Be sure to show **ALL** of your work.

    A) 336.2
    B) 354.0
    C) 381.9
    D) 442.4
    E) 487.4

15. Consider a 3-month \$41.50-strike American call option on a non-dividend paying stock. Suppose

    - the Black-Scholes framework holds;
    - the stock currently sells for \$40;
    - the stock's volatility is 30%;
    - the current value of Δ_{call} is 1/2;

    A) Find the continuously compounded risk-free interest rate r.

    B) Use your answer to part (A) to find the current price of the call option. Be sure to explain your reasoning.

16. You are given the following information for two European options on a stock priced using the Black-Scholes formula:

	45-strike put	50-strike call
Price	0.0211	10.2270
Δ	-0.0082	0.9300
Γ	0.0030	0.0160

The stock price is 60. Determine the number of shares of stock and the number of 50-strike calls one must buy or sell to Δ-Γ hedge a sale of a 45-strike put.

17. Consider a European call option on a stock following the Black-Scholes framework. The option expires in one year. Using the Black-Scholes formula for the option, you obtain

$$N(d_1) = 0.6591 \quad \text{and} \quad N(d_2) = 0.3409.$$

A) Find the volatility σ of the stock.

B) Use the concept of call elasticity Ω_{Call} to deduce the volatility of the option.

18. Let S_t be the price of a stock at time t. S_t follows a lognormal model and you are given the following information:

(i) The continuously compounded expected annual rate of return on the stock is 15%.

(ii) The continuously compounded dividend rate is 2%.

(iii) The stock's volatility is 30%.

(iv) $S_0 = 80$.

Calculate the probability that a 4-year European call with strike $K = 130$ has a payoff of 20 at expiration.

A) 0.227

B) 0.316

C) 0.392

D) 0.425

E) 0.504

19. Consider a European call option and a European put option on a non-dividend paying stock. The current stock price is \$60, the call option currently sells for \$0.15 more than the put option. Both the call and put option have strike price $K = \$70$ and will expire in 4 years. Find the continuously compounded annual risk-free interest rate r.

A) 3.92%
B) 4.22%
C) 4.79%
D) 5.01%
E) 5.66%

20. You are given the following information for a stock with price $S(t)$ at time t that follows the Black-Scholes framework:

(i) The time t price of a derivative security of the stock is $S(t)^a e^{0.0375t}$ with $a < 0$;
(ii) The continuously compounded risk-free interest rate is 0.05;
(iii) Neither the stock nor the derivative security pays dividends;
(iv) The volatility of the stock is 40%.

Use the Black-Scholes partial differential equation to determine a.

21. $X(t)$ follows the process

$$dX(t) = 0.5\,t\,dt + 0.2\,dW(t),$$

where $W(t)$ is standard Brownian motion. Another process $Y(t) = t\,e^{X(t)}$ satisfies

$$dY(t) = \alpha(t)\,Y(t)\,dt + \sigma(t)\,Y(t)\,dW(t).$$

Determine the lowest value of t for which $\alpha(t) = 2$.

A) 0.59
B) 0.65
C) 0.71
D) 0.77
E) 0.85

22. For two European call options on the same stock, C_1 and C_2:

- The stock price is 45.
- The price of C_1 is 10.86.
- The delta of C_1 is 0.6252.
- Mason buys one call of each type. Mason's portfolio has elasticity 2.90.
- Shay buys one C_1 and two C_2's. Shay's portfolio has elasticity 3.03.
- Sarah buys one C_1 and sells one C_2.

Determine the elasticity of Sarah's portfolio.

23. The time-t price of a stock is $S(t)$. You are given

- $S(t)$ follows a geometric Brownian motion.
- $S(0) = 1.2$.
- $\mathbb{P}(S(1) > 1.2) = 0.60642$.
- $\mathbb{P}(S(2) > 1.44) = 0.34827$.

Determine $\sigma^2_{S(1)}$.

A) 0.03
B) 0.04
C) 0.05
D) 0.06
E) 0.07