Math 458 - Financial Math for Actuaries II - Practice Exam # 2 - F23 - Solution

1. Consider a European call option on a stock following the Black-Scholes framework. The option expires in one year. Using the Black-Scholes formula for the option, you obtain

$$N(d_1) = 0.6591 \quad \text{and} \quad N(d_2) = 0.3409.$$

Find the volatility σ of the stock.

Solution: From the normal distribution table we know that

$$d_1 = N^{-1}(0.6591) = 0.41 \quad \text{and} \quad d_2 = N^{-1}(0.3409) = -0.41.$$

It follows from

$$d_2 = d_1 - \sigma \quad \text{that} \quad \sigma = 0.41 - (-0.41) = \boxed{0.82.}$$

2. A market-maker sells a 1-year European put option on a non-dividend paying stock and Δ-hedges it. You are given

- The current price of the stock is 50.
- The put option is at-the-money.
- The continuously compounded risk-free interest rate is 8%.
- The volatility of the stock is 22%.
- The option premium is 2.5632.
- The option Δ is -0.3179.

After 1 week, the stock price is 49. Determine the amount of money required to purchase additional shares to maintain the Δ-hedge.

A) -1.87

B) -1.85

C) -1.83

D) -1.81

E) -1.79

Solution: Since $\Delta = -0.3179$, it follows that Δ-hedging a portfolio consisting of 1 shorted put option requires shorting 0.3179 shares of the underlying stock. After one week (7 days), $\tau = 358/365$ and

$$d_1 = \frac{\ln\left(\frac{49}{50}\right) + \left(0.08 - 0 + 0.22^2/2\right) \cdot \tau}{\sigma\sqrt{\tau}} = 0.3763487... \simeq 0.38.$$

It follows that after 7 days, $\Delta_P = -N(-d_1) = -N(-0.38) = -0.3557$. Therefore the cost to maintain the hedge

$$49 \cdot (\Delta_P(49) - \Delta_P(50)) = 49 \cdot (-0.3557 - (-0.3179)) \simeq -1.8522.$$

Therefore, the correct answer is B.

Note: This solution assumes that d_1 is rounded to two significant digits and the normal distribution *table* is used to compute $N(-d_1) = 0.3557$. However, using exact values results in $\Delta_P(49) = -N(-d_1) = -N(-0.3763487) = -0.353329$ so that the cost to maintain the hedge is

$$49 \cdot (\Delta_P(49) - \Delta_P(50)) = 49 \cdot (-0.353329 - (-0.3179)) \simeq -1.73602.$$

The closest answer in the list of multiple choice options in this case would be **E**.

3. A stock has price 55, pays a continuous dividend rate of 2.5%, and has 20% continuous volatility. A portfolio Π has two European call options on this stock:

   - Call I allows purchase of 100 shares of the stock at the end of 3 months at a strike price of 55;
   - Call II allows purchase of 200 shares of the stock at the end of 6 months at a strike price of 60;

   If the risk-free continuous rate is 4%, calculate the elasticity of the portfolio.

   Solution: For call I we have $t = 1/4$ and $K = 55$ so that $d_1 \simeq 0.09$, $d_2 = d_1 - (0.2)\sqrt{1/4} \simeq -0.01$, and thus

$$C_I = S_0 e^{-\delta/4} N(d_1) - K e^{-r/4} N(d_2) = 55e^{-0.025/4}(0.5359) - 55e^{-0.04/4}(0.4960) \simeq 2.282.$$

   Thus, the elasticity of call I is

$$\Omega_I = \frac{S_0\Delta_I}{C_I} = \frac{55 \cdot e^{-0.025/4} \cdot (0.5359)}{2.282} \simeq 12.834.$$

Similarly, for call II we have $t = 1/2$ and $K = 60$ so that $d_1 =\simeq -0.49$, $d_2 = d_1 - (0.2)\sqrt{1/4} \simeq -0.63$, and therefore

$$C_{II} = S_0 e^{-\delta/2} N(d_1) - K e^{-r/2} N(d_2) = 55e^{-0.025/2}(0.3121) - 60e^{-0.04/2}(0.2643) \simeq 1.358.$$

Hence, the elasticity of call II is

$$\Omega_{II} = \frac{S_0\Delta_{II}}{C_{II}} = \frac{55 \cdot e^{-0.025/2} \cdot (0.3121)}{1.358} \simeq 12.48.$$

It follows that the elasticity of the portfolio $\Pi = \{100, 200\}$ is

$$\begin{aligned}
\Omega_\Pi &= \omega_I \Omega_I + \omega_{II}\Omega_{II}, \\
&= \left(\frac{100 \cdot C_I}{100 \cdot C_I + 200 \cdot C_{II}}\right) \cdot \Omega_I + \left(\frac{200 \cdot C_{II}}{100 \cdot C_I + 200 \cdot C_{II}}\right) \cdot \Omega_{II}, \\
&= \left(\frac{100 \cdot 2.282}{100 \cdot 2.282 + 200 \cdot 1.358}\right) \cdot 12.834 + \left(\frac{200 \cdot 1.358}{100 \cdot 2.282 + 200 \cdot 1.358}\right) \cdot 12.48, \\
&= \boxed{12.64.}
\end{aligned}$$

4. For two options on a non-dividend paying stock following the BSM framework, you are given:

Option	Δ	Γ	θ	Option Premium
1	0.5600	0.0800	-0.0156	3.00
2	0.3000	0.0320	-0.0068	1.00

In the above table, θ is expressed *per day* and the stock's price is 50. Determine r, the continuously compounded risk-free interest rate.

A) 0.043

B) 0.045

C) 0.048

D) 0.051

E) 0.054

Solution: Recall that $V_\tau = -\theta$, $V_S = \Delta$, and $V_{SS} = \Gamma$ so that the BSM PDE can be rewritten as

$$\frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV + 365\,\theta = 0.$$

The given data then yields two equations for σ^2 and r:

$$\overbrace{\frac{1}{2}\sigma^2 \cdot 50^2 \cdot (0.08)}^{\sigma^2 S^2 \Gamma/2} + \overbrace{r \cdot 50 \cdot (0.56)}^{rS\Delta} - \overbrace{r \cdot 3}^{rV} + \overbrace{365 \cdot (-0.0156)}^{365\,\theta} = 0,$$

$$\frac{1}{2}\sigma^2 50^2 (0.032) + r \cdot 50 \cdot (0.30) - r \cdot 1 + 365 \cdot (-0.0068) = 0.$$

Solving for σ^2 and r yields

$$r = 0.0511 \quad \text{and} \quad \sigma^2 = 0.044165.$$

The correct answer is D.

5. A stock's price follows a lognormal distribution. To simulate its price over 10 years, scenarios are generated. In each scenario, the stock price at time t is generated by generating a standard normal random variable Z. Then $S_t = S_{t-1} e^{0.1 + 0.2\,Z}$ for $t = 1, 2, \ldots, 10$. Find the expected value of the ratio S_{10}/S_0 in this simulation.

Solution: Let Z_i be the normal random variable in the i-th trial. Note that the Z_i are identically distributed and independent. Then

$$S_1 = S_0 e^{0.1+0.2\,Z_1}, \quad S_2 = S_1 e^{0.01+0.2\,Z_2} = S_0 e^{0.1+0.2\,Z_1} e^{0.1+0.2\,Z_2} = S_0 e^{0.1\cdot 2+0.2\,(Z_1+Z_2)}.$$

Continuing we have

$$S_{10} = S_0 e^{0.1\cdot 10+0.2(Z_1+\cdots Z_{10})} = S_0 e^{1+0.2\sum_{i=1}^{10} Z_i} \quad \Rightarrow \quad \frac{S_{10}}{S_0} = e \cdot e^{0.2\sum_{i=1}^{10} Z_i},$$

which is clearly lognormal. Recall that for $Y = e^X$ where $X \sim N\left(\mu,\, \sigma^2\right)$, $E[Y] = e^{\mu+\sigma^2/2}$. It follows from $E[0.2\,(Z_1 + \cdots + Z_{10})] = 0$ and $\sigma^2_{0.2(Z_1+\cdots+Z_{10})} = (0.04) \cdot \left[\sigma^2_{Z_1} + \cdots + \sigma_{Z^2_{10}}\right] = 0.04 \cdot 10 = 0.4$ that

$$E\left[\frac{S_{10}}{S_0}\right] = E\left[e \cdot e^{0.2\sum_{i=1}^{10} Z_i}\right] = e \cdot E\left[e^{0.2\sum_{i=1}^{10} Z_i}\right] = e \cdot e^{0+0.4/2} = \boxed{e^{1.2} \simeq 3.32.}$$

6. You are given:

- A stock has price 45.
- A market-maker writes put option I on a stock with price 1.53, $\Delta = -0.39$, and $\Gamma = 0.072$.
- The market-maker Δ-Γ hedges the option with the stock and with put option II having price 2.00, $\Delta = -0.31$, and $\Gamma = 0.038$.

Determine the number of shares of stock to buy to implement the hedge.

A) 0.19

B) 0.20

C) 0.21

D) 0.22

E) 0.23

Solution: The relevant portfolio is $\Pi = \{x_S,\, x_I,\, x_{II}\} = \{x_S,\, -1,\, x_{II}\}$ where x_S is the number of shares of stock and x_{II} is the number of option II puts. From the perspective of the market-maker, the value of the portfolio Π is

$$V_\Pi = x_1 \cdot S - P_I + x_{II} P_{II}.$$

To ensure that Π is Δ-Γ hedged we require

$$\Delta_\Pi = \frac{\partial V_\Pi}{\partial S} = x_S - \Delta_I + x_{II}\Delta_{II} = x_S - (-0.39) + x_{II}(-0.31) = x_S + 0.39 - 0.31x_{II} = 0$$

and

$$\Gamma_\Pi = \frac{\partial^2 V_\Pi}{\partial S^2} = -\Gamma_I + x_{II}\Gamma_{II} = -0.072 + 0.038x_{II} = 0 \quad \Rightarrow \quad x_{II} = \frac{0.072}{0.038} \simeq 1.89474.$$

Solving for x_S in the first equation thereby yields

$$x_S = 0.31x_{II} - 0.30 = 0.31(1.89474) - 0.39 = \boxed{0.197369.}$$

$\therefore$ **the correct answer is II.**

7. Let S_t be the price of a stock at time t. S_t follows a lognormal model and you are given the following information:

   - The continuously compounded expected annual rate of return on the stock is 15%.
   - The continuously compounded dividend rate is 2%.
   - The stock's volatility is 30%.
   - $S_0 = 80$.

   Calculate the probability that a 4-year European call with strike $K = 130$ has a payoff of 20 at expiration.

   A) 0.227

   B) 0.316

   C) 0.392

   D) 0.425

   E) 0.504

<u>Solution:</u> For a 4-year European call with strike $K = 130$ to have a payoff of at least 20 at expiration, we require that

$$(S_4 - 130)^+ \geq 20 \quad \text{or} \quad S_4 \geq 150.$$

Therefore, using the parameters in the problem, we need

$$\begin{aligned}
\mathbb{P}(S_4 \geq 150) &= \mathbb{P}\left(\ln(S_4) \geq \ln(150)\right), \\
&= \mathbb{P}\left(\ln(S_4) - \ln(80) \geq \ln(150) - \ln(80)\right), \\
&= \mathbb{P}\left(\ln\left(\frac{S_4}{80}\right) \geq \ln\left(\frac{150}{80}\right)\right), \\
&= \mathbb{P}\left(\ln\left(\frac{S_4}{80}\right) - (\alpha - \delta - \sigma^2/2)t \geq \ln\left(\frac{150}{80}\right) - (\alpha - \delta - \sigma^2/2)t\right), \\
&= \mathbb{P}\left(\frac{\ln\left(\frac{S_4}{80}\right) - (\alpha - \delta - \sigma^2/2)t}{\sigma\sqrt{t}} \geq \frac{\ln\left(\frac{150}{80}\right) - (\alpha - \delta - \sigma^2/2)t}{\sigma\sqrt{t}}\right), \\
&= \mathbb{P}\left(Z \geq \frac{\ln\left(\frac{150}{80}\right) - (\alpha - \delta - \sigma^2/2)t}{\sigma\sqrt{t}}\right), \\
&= \mathbb{P}\left(Z \geq \frac{\ln\left(\frac{150}{80}\right) - (0.15 - 0.02 - 0.3^2/2)(4)}{0.3\sqrt{4}}\right), \\
&= \mathbb{P}(Z \geq 0.4810), \\
&\simeq 1 - \mathbb{P}(Z < 0.48), \\
&= 1 - 0.6844, \\
&= \boxed{0.3156.}
\end{aligned}$$