Math 458 - Financial Math for Actuaries II - Practice Exam # 1 - Solution

1. (20 total points - 2 points each) Please circle either T (true) or F (false) for each of the below statements. All prices are no-arbitrage prices. **Answers are in BOLD.**

I) **T** F Combining a short straddle with strike price $K_2 = \$60$ with a long strangle with strikes $K_1 = 30$ and $K_3 = 90$ forms a symmetric butterfly spread.

II) **T** F If $r = \delta$ for a stock with price S_t, then the forward price $F_{0,T}$ is equal to the asset price S_0.

III) T **F** It is never advantageous to exercise an American call option early.

IV) T **F** Given 1-year European puts with risk-free rate is $r = 10\%$, $\delta = 5\%$, and $\sigma = 20\%$, it is possible for the arbitrage-free price $P^E(K)$ to satisfy

$$P^E(30) > P^E(10) + 20e^{0.25}.$$

V) T **F** In the Cox-Ross-Rubenstein binomial model, if $u = 1.2$ then $d = 0.8$.

VI) **T** F In a 5-period binomial tree with initial stock price $S_0 = \$100$, expiration $T = 1$ year, $u = 1.1$, and $d = 0.9$, the second highest value of S_1 is \$131.769.

VII) **T** F Given a risk-free rate $r > 0$ and continuous dividend rate $\delta > 0$, one way to compute the risk-neutral probability $\tilde{p}$ for the binomial model is to require that the risk-neutral expected value of the stock S_{t+1} is equal to the forward price of the stock. That is, solve

$$\tilde{E}[S_{t+1}] = \tilde{p}(uS_t) + (1 - \tilde{p})(dS_t) = S_t e^{(r-\delta)h}.$$

for $\tilde{p}$.

VIII) T **F** In a 3-period binomial tree where the probability that the stock moves "up" is p, the probability that the stock has two "up" moves and one "down" move, in any order, is $p^2(1 - p)$.

IX) T **F** Because the Black-Scholes formula gives the arbitrage-free price of a call option in continuous time, you cannot use it to price a call on a stock where only discrete dividends are paid (e.g., owning a stock pays dividends only twice/year).

X) **T** F For a 6-month European option, modeled using the Black-Scholes-Merton framework, $d_1 = d_2$ requires $\sigma = 0$.

2. (20 points) Consider a European call option and a European put option on a non-dividend paying stock. The current stock price is \$60, the call option currently sells for \$0.15 more than the put option. Both the call and put option have strike price $K = \$70$ and will expire in 4 years. Find the continuously compounded annual risk-free interest rate r.

Solution: Put-Call Parity (PCP) states for a non-dividend paying stock that

$$C^E(K) - P^E(K) = PV\left[F_{0,T} - K\right] = e^{-rT}\left(S_0 e^{rT} - K\right) = S_0 - Ke^{-rT}.$$

Since $S_0 = 60$, $T = 4$, and

$$C^E(70) - P^E(70) = 0.15,$$

it follows that we just need to solve

$$0.15 = 60 - 70e^{-4r}$$

for r. Clearly,

$$70e^{-4r} = 60 - 0.15 = 59.85 \quad \Rightarrow \quad r = -\frac{1}{4}\ln\left(\frac{59.85}{70}\right) \simeq 0.03916345\ldots$$

or

$$\boxed{r \simeq 3.916\%}$$

3. (20 points) It is known that for a European put option, the arbitrage-free price $P^E(K)$ satisfies

$$P^E(20) = 55 \quad \text{and} \quad P^E(110) = 125.$$

A) (14 points) Use convexity to find a minimum value of $P(155)$.

Solution: Letting $K_1 = 20$, $K_2 = 110$, and $K_3 = 155$, solving

$$\lambda K_1 + (1-\lambda)K_3 = 20\lambda + (1-\lambda)155 = 155 - 135\lambda = K_3 = 110$$

we have

$$\boxed{\lambda = \frac{155-110}{135} = \frac{1}{3}.}$$

It follows from convexity that

$$125 = P^E(K_2) = P^E\left(\frac{1}{3}K_1 + \left(1 - \frac{1}{3}\right)K_3\right) \leq \frac{1}{3}P^E(K_1) + \frac{2}{3}P^E(K_3) = \frac{1}{3}(55) + \frac{2}{3}P^E(155)$$

Solving for $P^E(K_3)$ yields

$$3(125) \leq 55 + 2P^E(155) \quad \Rightarrow \quad P^E(155) \geq \frac{375-55}{2} = \frac{320}{2} = 160$$

so that

$$\boxed{P^E(155) \geq 160.}$$

B) (6 points) Find an upper bound to $P(155)$ and compare it to your answer in (A).

<u>*Solution:*</u> Here we use the inequality

$$K \leq L \quad \Longrightarrow \quad P^E(K) - P^E(L) \leq K - L.$$

- When $K = 20$ and $L = 155$ we have

$$P^E(155) - P^E(20) \leq 155 - 20 = 135 \quad \Rightarrow \quad P(155) \leq P^E(20) + 135 = 55 + 135 = 190.$$

- When $K = 110$ and $L = 155$ we have

$$P^E(155) - P^E(110) \leq 155 - 110 = 45 \quad \Rightarrow \quad P^E(155) \leq P^E(110) + 45 = 125 + 45 = 170.$$

It follows that $P^E(155)$ must satisfy

$$\boxed{160 \leq P^E(155) \leq 170}$$

4. (20 total points) MSU stock is currently worth \$60 a share. The continuously compounded risk-free rate is $r = 6\%$, the annual continuously compounded dividend yield is $\delta = 2\%$, and the annualized standard deviation of the continuously compounded stock return is $\sigma = 15\%$.

A) (6 points) Using the Cox-Ross-Rubenstein model, find up and down factors u and d for a 2-period binomial model over the course of one year.

<u>*Solution:*</u> Because $h = 1/2$ we have

$$u = e^{\sigma\sqrt{h}} = e^{(0.15)/\sqrt{2}} \simeq 1.111895 \quad \text{and} \quad d = e^{-\sigma\sqrt{h}} = \frac{1}{u} \simeq 0.899365$$

so that

$$\boxed{u \simeq 1.111895 \quad \text{and} \quad d \simeq 0.899365.}$$

B) (4 points) Use your answer to part (A) to draw the 2-period stock price tree.

<u>*Solution:*</u> Using

$$S_{t+1}^{-} = S_t d \quad \text{and} \quad S_{t+1}^{+} = S_t u$$

for $t = 0$ and $t = 1$ it follows that

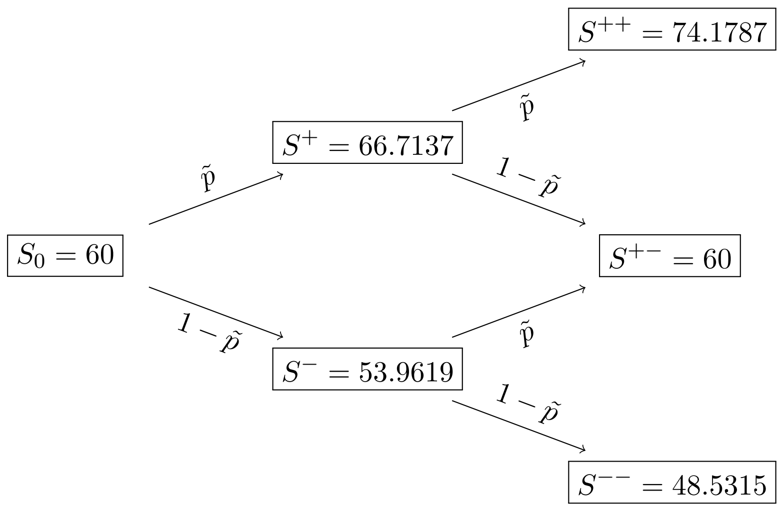

Figure 1: Binomial model price tree for #4.

C) (10 points) Use your answers to parts (A) & (B) to find the arbitrage free price of a 1-year European derivative with payoff at expiration given by

$$\Lambda(S) := \left\{ \begin{array}{ccc} 0 & : & S < 55 \\ 10 & : & 55 \leq S \leq 70 \\ 0 & : & S > 70 \end{array} \right\}.$$

<u>*Solution:*</u> At expiration, the value of the European option is $V(S_T) = \Lambda(S_T)$. Using the price tree from part (B) we have

$$\Lambda(S_T = 48.5315) = \Lambda(S_T = 74.1787) = 0 \quad \text{and} \quad \Lambda(S_T = 60) = 10.$$

The risk-neutral probability for this binomial model is

$$\tilde{p} = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(.06-.02)/2} - 0.899365}{1.111895 - 0.899365} \simeq 0.56856.$$

Therefore, letting $h = 1/2$ we have

$$V^E\left(S_{1/2}^{+} = 66.7137\right) = e^{-rh}\tilde{E}[S_1] = e^{-rh}(\tilde{p}(0) + (1-\tilde{p})60) \simeq 4.18689$$

and

$$V^E\left(S_{1/2}^{-} = 66.7137\right) = e^{-rh}\tilde{E}[S_1] = e^{-rh}(\tilde{p}(60) + (1-\tilde{p})0) \simeq 5.517565.$$

Finally, the arbitrage-free price of the derivative at time $t = 0$ is

$$\boxed{V^E(S_0) = e^{-rh}\left(\tilde{p}(4.18689) + (1-\tilde{p})(5.517565)\right) \simeq 4.62029.}$$

Putting this solution in "tree-form" yields

$$\begin{tikzcd}
 & & \boxed{V\left(S^{++}\right) = 0} \\
 & \boxed{V\left(S^{+}\right) = 4.18689} \arrow[ur, "\tilde{p}"] \arrow[dr, "1-\tilde{p}"] & \\
\boxed{V(S_0) = 4.62029} \arrow[ur, "\tilde{p}"] \arrow[dr, "1-\tilde{p}"'] & & \boxed{V\left(S^{+-}\right) = 10} \\
 & \boxed{V\left(S^{-}\right) = 5.517565} \arrow[ur, "\tilde{p}"] \arrow[dr, "1-\tilde{p}"] & \\
 & & \boxed{V\left(S^{--}\right) = 0}
\end{tikzcd}$$

Figure 2: Derivative prices for binomial model in #4.

5. (20 points) You are considering the purchase of 100 units of a 3-month 25-strike European put option on a stock. You know that

- The Black-Scholes framework holds.
- The stock currently sells for \$20.
- The stock's volatility is 24%.
- The stock pays dividends continuously with a dividend yield of 3%.
- The continuously compounded risk-free interest rate is 5%.

Find the price of a block of 100 of these put options. Be sure to show **ALL** of your work.

Solution: Given

$$S_0 = \$20, \quad T = \frac{1}{4}, \quad K = \$25, \quad r = 0.05, \quad \delta = 0.03, \quad \text{and} \quad \sigma = 0.24$$

it follows that

$$d_1 = \frac{\ln\left(\frac{20}{25}\right) + \left(0.05 - 0.03 + \frac{(0.24)^2}{2}\right)\left(\frac{1}{4}\right)}{0.24\sqrt{\frac{1}{4}}} \simeq -1.75786 \quad \text{and} \quad d_2 = d_1 - \frac{0.24}{\sqrt{4}} \simeq -1.87786.$$

Therefore, rounding $d_1 \doteq -1.76$ and $d_2 \doteq -1.88$ the cumulative normal distribution function table gives

$$N(-d_1) = N(1.76) = \int_{-\infty}^{1.76} \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\, d\xi = 0.9608 \quad \text{and} \quad N(-d_2) = N(1.88) = 0.9699.$$

Hence, the Black-Scholes formula for the price of a put yields

$$\begin{aligned} P^E &= Ke^{-rT}N(-d_2) - S_0e^{-\delta T}N(-d_1), \\ &= 25e^{-.05/4}(0.9699) - 20e^{-0.03/4}(0.9608), \\ &= 4.87387. \end{aligned}$$

Therefore, the arbitrage free price of 100 puts is

$$\boxed{100P^E = \$487.387.}$$

6. (20 points) A stock price S_t that follows the BSM framework, you know $S_0 = 40$. Further, you are given

- The continuously compounded expected rate of return on the stock is 12%.
- The continuously compounded dividend yield is 2%.
- The continuously compounded risk-free interest rate is 3%.

European call and put options expiring in one year have strike price 50. The expected payoff on the call option is 4.40. Find the expected payoff of the put option.

I) 9.89
II) 10.19
III) 11.95
IV) 13.71
V) 14.40

<u>*Solution:*</u> Construct a synthetic forward by shorting the put and buying the call. Then, since the strike price is $K = 50$, the guaranteed payoff under all possible values of S_1 will be $S_1 - 50$. Therefore the expected payoff of the call option minus the put option is

$$\begin{aligned} E[(S_1-50)^+ - (50-S_1)^+] &= \overbrace{E[(S_1-50)^+]}^{\text{expected call payoff}} - \overbrace{E[(50-S_1)^+]}^{\text{expected put payoff}}, \\ &= 4.40 - E[(50-S_1)^+] = E[S_1 - 50] = E[S_1] - 50. \end{aligned}$$

Now, since

$$E[S_1] = S_0 e^{(\alpha-\delta)}T = 40 \cdot e^{(0.12-0.02)\cdot 1} \simeq 44.2068,$$

it follows that

$$E[(50-S_1)^+] = 50 + 4.40 - 44.2068 = 10.1932 \simeq \boxed{10.1932.}$$

$\therefore$ **The correct answer is II.**