Lecture 31 – April 22, 2014

Inst. Mark Iwen — *Scribe: Atreyee Majumder*

1 Nonequispaced DFT Methods

Goal: Rapidly compute $f(x) := \sum_{\omega=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}(\omega) e^{2\pi i \omega x}$ at $M = O(N)-$ values $x_1, x_2, \dots x_M \in [0,1)$.

- The values $x_1 \dots x_M$ are not exactly $\frac{j}{N}$ for $j \in [N]$ so we cannot directly use an FFT.
- Naive method takes $O(N^2)-$ floating point operations. We want to do it faster.

1.1 Method 1 (see [1])

Let $\theta : [0,1] \to \mathbb{C}$ be periodic with an absolutely convergent Fourier series.

For example, assume that:

1. $\hat{\theta}(k) := \int_0^1 \phi(x) e^{-2\pi i k x} dx, k \in \mathbb{Z}$ decays like $|\hat{\theta}(k)| < \epsilon. \max\left\{1, \left||k| - \frac{N}{2}\right|^{-m}\right\}$, for some $m > 1$ $\forall k \in \mathbb{Z} \setminus \left[-\frac{N}{2}, \frac{N}{2}\right)$, and
2. $|\hat{\theta}(k)| > \epsilon, \forall k \in \left[-\frac{N}{2}, \frac{N}{2}\right)$

for some $\epsilon \in \mathbb{R}^+$ (not too small!). We can then set

- Set $\hat{c}_\omega := \begin{cases} \frac{\hat{f}(\omega)}{\hat{\theta}(\omega)} & \text{if } \omega \in \left[-\frac{N}{2}, \frac{N}{2}\right) \\ 0 & \text{otherwise} \end{cases}$
- Choose $\alpha > 1$ such that $\alpha N \in \mathbb{N}$ is even.
- Compute $c_l := \frac{1}{\alpha N} \sum_{\omega=-\frac{\alpha N}{2}}^{\frac{\alpha N}{2}-1} \hat{c}_\omega e^{\frac{2\pi i l \omega}{\alpha N}}, l \in [\alpha N]$. We have $\sum_l |c_l| \leq \frac{\|\hat{f}\|_1}{\alpha N \epsilon}$
- Computing $c_l, \forall l \in [\alpha N]$ takes $O(\alpha N \cdot \log(\alpha N))-$ time using the FFT. We have that

$$f(x) = \sum_{\omega=-\frac{\alpha N}{2}}^{\frac{\alpha N}{2}-1} \frac{\hat{f}(\omega)}{\hat{\theta}(\omega)} \hat{\theta}(\omega) e^{2\pi i \omega x} \qquad \left(\text{set } \hat{f}(\omega) = 0 \text{ for } \omega \notin \left[-\frac{N}{2}, \frac{N}{2}\right)\right)$$

$$= \sum_{\omega} \frac{\hat{f}(\omega)}{\hat{\theta}(\omega)} \left[\sum_{p=-\frac{\alpha N}{2}}^{\frac{\alpha N}{2}-1} \hat{\theta}(p) e^{2\pi i p x} \left(\frac{1}{\alpha N} \sum_{l=0}^{\alpha N-1} e^{\frac{2\pi i l(\omega-p)}{\alpha N}} \right) \right] \qquad \left(\frac{1}{\alpha N} \sum_{l=0}^{\alpha N-1} e^{\frac{2\pi i l(\omega-p)}{\alpha N}} = \delta(\omega - p) \right)$$

$$
\begin{aligned}
&= \frac{1}{\alpha N}\sum_{\omega}\frac{\hat{f}(\omega)}{\hat{\theta}(\omega)}\left[\sum_{l}\sum_{p}\hat{\theta}(p)e^{2\pi i p\left(x-\frac{l}{\alpha N}\right)}e^{\frac{2\pi i l\omega}{\alpha N}}\right] \\
&= \frac{1}{\alpha N}\sum_{\omega}\frac{\hat{f}(\omega)}{\hat{\theta}(\omega)}\sum_{l}e^{\frac{2\pi i l\omega}{\alpha N}}\left[\theta\left(x-\frac{l}{\alpha N}\right)+O\left(\epsilon\cdot\int_{\frac{\alpha N}{2}}^{\infty}\left(k-\frac{N}{2}\right)^{-m}dk\right)\right] \\
&= \sum_{l=0}^{\alpha N-1}\frac{1}{\alpha N}\left(\sum_{\omega}\frac{\hat{f}(\omega)}{\hat{\theta}(\omega)}e^{\frac{2\pi i l\omega}{\alpha N}}\right)\left[\theta\left(x-\frac{l}{\alpha N}\right)+O\left(\frac{\epsilon}{m-1}\left(\frac{(\alpha-1)N}{2}\right)^{1-m}\right)\right] \\
&= \sum_{l=0}^{\alpha N-1}c_l\left[\theta\left(x-\frac{l}{\alpha N}\right)+O\left(\frac{\epsilon}{m-1}\left(\frac{(\alpha-1)N}{2}\right)^{1-m}\right)\right] \\
&= \sum_{l=0}^{\alpha N-1}c_l\cdot\theta\left(x-\frac{l}{\alpha N}\right)+O\left(\frac{\|\hat{f}\|_1}{\alpha N(m-1)}\left(\frac{(\alpha-1)N}{2}\right)^{1-m}\right) \qquad (1)
\end{aligned}
$$

This expression is **useful** if θ is a nice function that

i. has m large (s.t. it decays nicely in Fourier), and

ii. also has $\theta(x)$ decay very quickly away from 0.

- One good choice is to take

$$
\theta(\nu) := (\Pi b)^{-1/2}\sum_{r\in\mathbb{Z}}e^{-(\alpha N(\nu+r))^2/b},\ \text{for } b\in\mathbb{R}^+
$$

  and then we approximate $\theta(\nu)$ with

$$
\psi(\nu) := (\Pi b)^{-1/2}\sum_{r\in\mathbb{Z}}e^{-(\alpha N(\nu+r))^2/b}\mathcal{X}_{[-n,n]}(\alpha N(\nu+r))
$$

  for $1\le b\le\frac{2\alpha n}{(2\alpha-1)\pi}$, where $n\ll N$ is small.

- Making this choice gives an error $O\left(\|\hat{f}\|_1 e^{-b\pi^2\left(1-\frac{1}{\alpha}\right)}\right)$.

1.1.1 Wrap Up

Decide once on a function θ, and then compute $\hat{\theta}$ and ψ "once". Pick $\alpha\in\mathbb{R}^+$. Then,
Given: $f(x)=\sum_{\omega}\hat{f}(\omega)e^{2\pi i\omega x}$ and $x_1\dots x_M$.

- Compute weights $c_l=\frac{1}{\alpha N}\sum_{\omega=-\frac{\alpha N}{2}}^{\frac{\alpha N}{2}-1}\left(\frac{\hat{f}(\omega)}{\hat{\theta}(\omega)}\right)e^{\frac{2\pi i l\omega}{\alpha N}},\forall l\in[\alpha N]$.
- Set $\tilde{f}(x_j)=\sum_{l=-\log(\delta^{-1})}^{\log(\delta^{-1})}c_{\lfloor\alpha N x_j\rfloor+l}\cdot\psi\left(x_j-\frac{\lfloor\alpha N x_j\rfloor+l}{\alpha N}\right),\forall j\in[M]+1$
- Each $|f(x_j)-\tilde{f}(x_j)|\le O\left(\|\hat{f}\|_1\cdot\delta^{\pi^2\left(1-\frac{1}{\alpha}\right)}\right)$
- Total runtime of 2- step procedure is $O\left(\alpha N\log(\alpha N)+N\log\left(\frac{1}{\delta}\right)\right)$

1.2 Method 2 (Another approach) [2]

- Given: $f(x) = \sum_{\omega=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}(\omega)e^{2\pi i\omega x}$, we again want to compute f at $x_1 \dots x_M \in [0,1)$.
- Choose $M = 2^q \geq \frac{\pi N}{2}$ (power of 2 for FFT efficiency).
- Let $y_j = \frac{j}{M}$ for $j \in [M]$.
- Note that $f^{(1)}(x) = \sum_\omega \hat{f}(\omega)(2\pi i\omega)e^{2\pi i\omega x}$
  $f^{(2)}(x) = \sum_\omega \hat{f}(\omega)(2\pi i\omega)^2 e^{2\pi i\omega x}$
  $\vdots$
  $f^{(l)}(x) = \sum_\omega \hat{f}(\omega)(2\pi i\omega)^l e^{2\pi i\omega x}$
- Compute $f^{(l)}(y_j)$ for all $l \in [p], j \in [M]$ in $O(lN \log N)-$ time via $l-$ FFTs.
- Taylor's Theorem tells us that $f(x_j) = \sum_{l=0}^{p-1} f^{(l)}(\tilde{y}_j)\frac{(x_j-\tilde{y}_j)^l}{l!} + \frac{f^{(p)}(\xi)}{p!}(x_j - \tilde{y}_j)^p$ where $\tilde{y}_j$ is the point y_j closest to x_j, and ξ lies between x_j and $\tilde{y}_j$.
- Note that, the error term is $\frac{\|\hat{f}\|_1\left(\frac{N}{2}\right)^p(2\pi)^p\left(\frac{1}{\pi N}\right)^p}{p!} \leq \frac{\|\hat{f}\|_1}{p!}$
  $\implies p = O(\log N)$ suffices for decent accuracy most days.
  Therefore, $O(N \log^2 N)-$ time gives "decent accuracy".

If the frequencies, ω, are not integers – but we want to evaluate the sums at equally spaced points – we can use essentially the same techniques to compute the sums rapidly.

References

[1] Daniel Potts, Gabriele Steidl, Manfred Tasche. Fast Fourier transforms for nonequispaced data: A tutorial. *Modern sampling theory. Birkhuser Boston*, 2001.

[2] Chris Anderson, Marie Dillon Dahleh. Rapid computation of the discrete Fourier transform. *SIAM Journal on Scientific Computing 17.4*, 1996.