MTH 995-003: Intro to CS and Big Data Spring 2014

Lecture 23 – March 25, 2014

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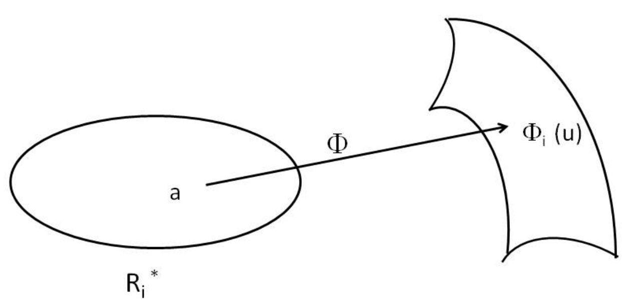
1 Homework: Reach and Projections onto Manifolds

- Let $\mathcal{M} \subseteq \mathbb{R}^D$ be a d-dimensional manifold. We define the projection onto $\mathcal{M}$ by

$$\Pi_{\mathcal{M}}(\vec{x}) := \text{ the unique point } \vec{y} \in \overline{\mathcal{M}} \text{ with} \|\vec{x} - \vec{y}\|_2 = d(\vec{x}, \overline{\mathcal{M}}) \quad \forall \vec{x} \in D(\mathcal{M})$$

- Let $\vec{x} \in \text{tube}_\tau(\mathcal{M})$ where $\tau := \text{reach}(\mathcal{M})$. Furthermore, let $\text{r} := \text{d}(\vec{x}, \mathcal{M}) < \tau$.
- Suppose that $\Pi_{\mathcal{M}}(\vec{x}) = \vec{m} \in$ the interior of $\mathcal{M}$.

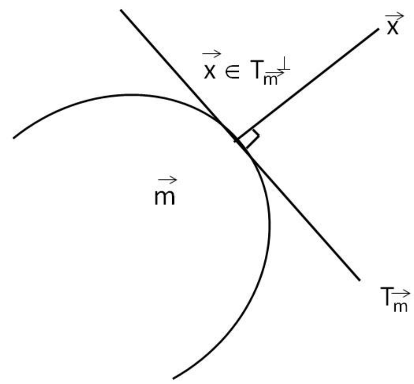
Definition 1. *The interior of $\mathcal{M}$ is the set of points $\vec{z} \in \mathcal{M}$ such that $\exists \vec{u} \in$ the interior of $R_i^\star$, for some $i \in I$, with the property that $\Phi_i(\vec{u}) = \vec{z}$.*



1.1 Homework Problems (Due April 3, 2014)

Given the above setup, show the following:

1. See Lecture 21, section 1.2
2. Show that $\exists \vec{v} \in T_{\vec{m}}^{\perp}$, $\|\vec{v}\|_2 = 1$, such that $\vec{x} = \vec{m} + r \cdot \vec{v}$



3. Show that $\vec{v}$ from #2 does not depend on r, i.e., show that $\Pi_{\mathcal{M}}(\vec{m}+\tilde{r}\cdot\vec{v}) = \vec{m}$ for all $\tilde{r} \in [0, r]$.
4. Let $\vec{z} \in \overline{\mathcal{M}}$. Show that $\Pi_{\mathcal{M}}^{-1}(\vec{z}) \cap \text{tube}_{\tau}(\mathcal{M}) \subseteq \mathbb{R}^D$ is convex.
5. Let $\vec{m} \in$ interior($\mathcal{M}$), let $\vec{v} \in \mathrm{T}_{\vec{m}}^{\perp}$ with $\| \vec{v} \|_2 = 1$ and let $r \in [0,\tau)$.
   Show that $\Pi_{\mathcal{M}}(\vec{m} + r \cdot \vec{v}) = \vec{m}$.

Hint for #5: Consider

$$\epsilon(\vec{v}) := \sup \left\{ r \in \mathbb{R}^+ \middle| \Pi_{\mathcal{M}}(\vec{m} + r\vec{v}) = \vec{m} \right\}.$$

Show that $\epsilon(\vec{v}) > 0 \implies \epsilon(\vec{v}) \geq \text{reach}(\mathcal{M})$. Then, argue that $\epsilon(\vec{v}) \neq 0$. You can waive your hands a little.

• This last question shows that "the open normal bundle about $\mathcal{M}$ of radius τ",

$$\left\{ \left(\vec{m} + T_{\vec{m}}^{\perp} \right) \cap B_{\tau}(\vec{m}) \;\middle|\; \vec{m} \in \text{interior}(\mathcal{M}) \right\},$$

is a subset of $\Pi_{\mathcal{M}}^{-1}(\text{interior}(\mathcal{M}))$.

2 Toward Covering Numbers for Compact Manifolds

Recall the end of Lecture 22: We we want a lower bound for the d-dimensional volume

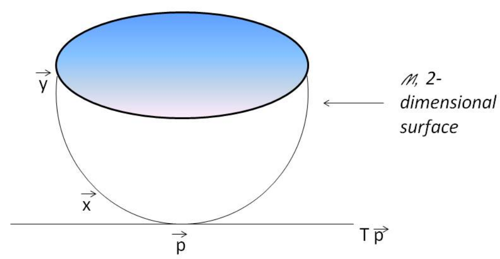
$$\text{vol}(B_r(\vec{p}) \cap \mathcal{M})$$

that holds for "most" $\vec{p} \in \mathcal{M}$. For this lower bound we need a couple of lemmas.

Lemma 1. *Let $\mathcal{M} \subseteq \mathbb{R}^D$ be a d-dimensional manifold with $\tau := \text{reach}(\mathcal{M}) > 0$. Let $\vec{p} \in \mathcal{M}$ and $T_{\vec{p}}$ be the d-dimensional tangent space to $\mathcal{M}$ at $\vec{p}$. Then $\Pi_{T_{\vec{p}}}$ is invertible on $\mathcal{M} \cap B_{\tau/4}(\vec{p})$.*

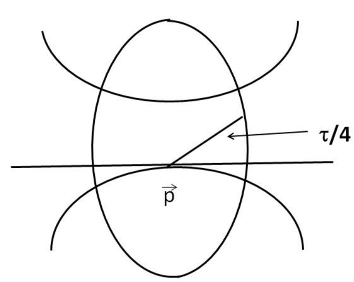
Proof: See [2] and [1]. □

Proof Idea: Let $\vec{x}, \vec{y} \in \mathcal{M} \cap B_{\tau/4}(\vec{p})$ and consider a geodesic path from $\vec{x}$ to $\vec{y}$.



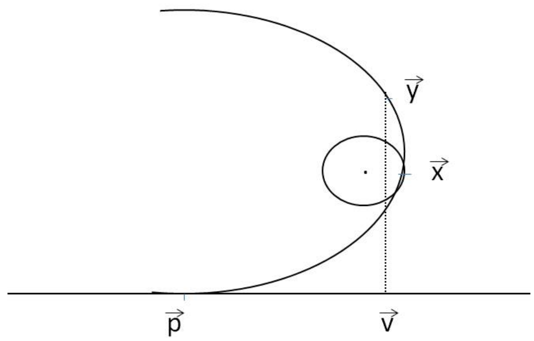
What could go wrong?

1.) $\mathcal{M}$ has more than one component in $B_{\tau/4}(\vec{p})$, and $\vec{x}$ and $\vec{y}$ are not on the same component.



$\Longrightarrow$ impossible by the definition of the reach.

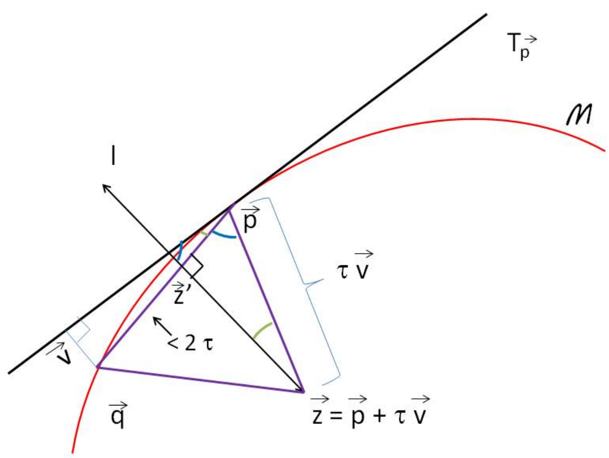
2.) $\vec{x}$ and $\vec{y}$ are on a path that is entirely contained in $B_{\tau/4}(\vec{p})$. Then $\Pi_{T_{\vec{p}}}(\vec{x}) = \Pi_{T_{\vec{p}}}(\vec{y})$ means



$\implies$ impossible because the curvature of the path cannot be so high compared to τ.

Lemma 2. *Let $\vec{p}, \vec{q} \in \mathcal{M}$ be such that $\|\vec{p}-\vec{q}\|_2 < 2 \cdot \tau := 2 \cdot \text{reach}(\mathcal{M})$. Then, the angle between $(\vec{p}-\vec{q})$ and $\Pi_{T_{\vec{p}}}(\vec{p}-\vec{q}) \leq \sin^{-1}\left(\frac{\|\vec{p}-\vec{q}\|_2}{2\tau}\right)$.*

Proof: Let $\vec{v}$ be a unit vector along $(I - \Pi_{T_{\vec{p}}})(\vec{p}-\vec{q}) \in T_{\vec{p}}^{\perp}$ and set $\vec{z} = \vec{p} + \tau \cdot \vec{v}$



- Note that $\| \vec{z} - \vec{q} \|_2 \geq \tau$ since $\vec{z} \in \overline{D(\mathcal{M})} \cap T_{\vec{p}}^{\perp}$ by HW #5 above.
- Let l be the line through $\vec{z}$ that is perpendicular to $(\vec{p}-\vec{q})$
- Let $\vec{z}\,'$ be the intersection of l and $(\vec{p}-\vec{q})$

• Note that $\angle \vec{z}\,\vec{p}\,\vec{q} \leq 90°$ (by construction of $\vec{z}$)

• $\angle \vec{z}\,\vec{q}\,\vec{p} \;\leq\; \angle \vec{z}\,\vec{p}\,\vec{q} \;\leq\; 90°$ $\qquad (\dagger)$

since $\| \vec{z} - \vec{q} \|_2 \geq \| \vec{z} - \vec{p} \|_2$

• Hence, $\vec{z}\,'$ is in between $\vec{p}$ and $\vec{q}$ (i.e., our picture is accurate). Also, more importantly, ($\dagger$) implies that $\| \vec{z}\,' - \vec{p} \|_2 \leq \frac{1}{2} \; \| \vec{p} - \vec{q} \|_2$

$\implies \sin(\angle \vec{p}\,\vec{z}\,\vec{z}') \;\leq\; \frac{\|\vec{p}-\vec{q}\|_2}{2\,\tau}$ $\square$

We can now obtain the desired lower bound.

Lemma 3. *Let* $\vec{p} \in \text{interior}(\mathcal{M})$, $\tau = \text{reach}(\mathcal{M})$ *and* $r \in [0, \tau/4)$. *Then,*

$$\text{vol}(B_r(\vec{p}) \cap \mathcal{M}) \geq \left(1 - \frac{r^2}{4\,\tau^2}\right)^{\frac{d}{2}} \cdot r^d \cdot \text{vol}\left(\text{unit ball in } \mathbb{R}^d\right)$$

$\forall\, \vec{p}$ *s.t.* $B_r(\vec{p}) \cap \text{boundary}(\mathcal{M}) = \emptyset$.

Proof: We will show $\exists r\,' \;\geq\; \sqrt{1 - \frac{r^2}{4\,\tau^2}} \cdot r$ such that

$$(*) \qquad\qquad B_{r'}(\vec{p}) \cap T_{\vec{p}} \;\subset\; \Pi_{T_{\vec{p}}}(B_r(\vec{p}) \cap \mathcal{M})$$

Note: $(*)$ implies the desired bound since

$$\begin{aligned}
&\text{vol}(B_r(\vec{p}) \cap \mathcal{M}) \\
&\qquad \geq \text{vol}\left(\Pi_{T_{\vec{p}}}(B_r(\vec{p}) \cap \mathcal{M})\right) && \text{(projections are non-expansive)} \\
&\qquad \geq \;\text{vol}(B_{r'}(\vec{p}) \cap T_{\vec{p}}) && \text{(by (*))} \\
&\qquad = (r')^d \cdot \text{vol (unit ball in } T_{\vec{p}})
\end{aligned}$$

a + b < length of the curve

Thus, it suffices to show $(*)$. We will do this in the next lecture.

References

[1] Partha Niyogi, Stephen Smale, Shmuel Weinberger. Finding the Homology of Submanifolds with High Confidence from Random Samples. *J. Discrete Comput. Geom.*, 39(1): 419–441, 2008.

[2] Armin Eftekhari, Michael B. Wakin. New Analysis of Manifold Embeddings and Signal Recovery from Compressive Measurements. *J. CoRR* , 1306.4748, 2013.