Lecture 22 — Mar 20th, 2014

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1 Further Background on Manifolds

Definition 1. *For a d-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^D$ with atlas $(\Phi_i, R_i^*)_{i \in I}$, we define*

$$D(\mathcal{M}) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^D : \textit{there is a unique } \mathbf{y} \in \overline{\mathcal{M}} \textit{ with } \|\mathbf{x} - \mathbf{y}\|_2 = d(\mathbf{x}, \overline{\mathcal{M}})\},$$

where

$$\overline{\mathcal{M}} = \bigcup_{i \in I} \Phi_i(\overline{R_i^*})$$

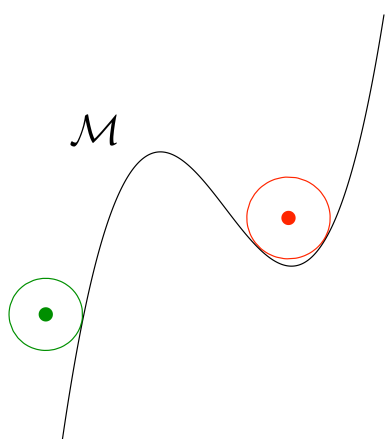
and $d(\cdot, \cdot)$ is the Euclidean distance

$$d(\mathbf{x}, \overline{\mathcal{M}}) = \inf_{\mathbf{y} \in \overline{\mathcal{M}}} \|\mathbf{x} - \mathbf{y}\|_2.$$

Thus, $\overline{\mathcal{M}} \subseteq D(\mathcal{M})$ and for $\mathbf{x} \notin \overline{\mathcal{M}}$,

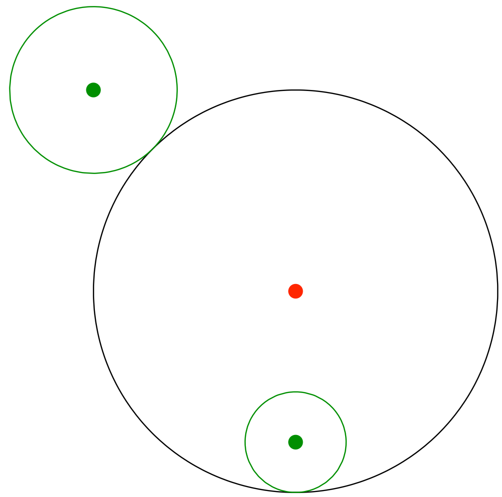
$$\mathbf{x} \in D(\mathcal{M}) \iff \text{there is } r \in \mathbb{R}^+ \text{ such that } |\overline{B_r(\mathbf{x})} \cap \overline{\mathcal{M}}| = 1.$$

For example, in the figure below, the point in red (on right) is not an element of $D(\mathcal{M})$, while the point in green (on left) is.



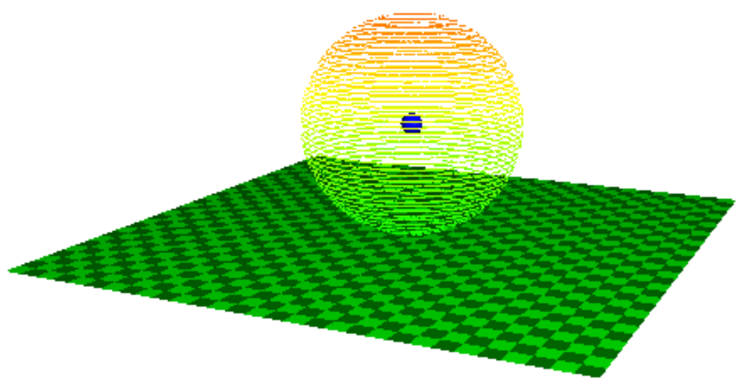
More concretely, suppose $\mathcal{M}$ is the sphere of radius r in $\mathbb{R}^D$, centered at $\mathbf{c}$. Then

$$D(\mathcal{M}) = \mathbb{R}^D \setminus \{\mathbf{c}\}.$$



If $\mathcal{M}$ is a d-dimensional affine subspace of $\mathbb{R}^D$, then

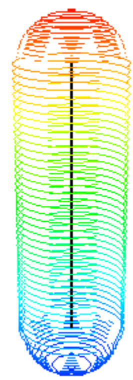
$$D(\mathcal{M}) = \mathbb{R}^D.$$



Definition 2. *For a d-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^D$, the **tube** of radius $r \in \mathbb{R}^+$ around $\mathcal{M}$ is*

$$\mathsf{tube}_r(\mathcal{M}) \stackrel{\mathsf{def}}{=} \{\mathbf{x} \in \mathbb{R}^D : d(\mathbf{x}, \mathcal{M}) < r\}.$$

For example, if $\mathcal{M}$ is a line segment in $\mathbb{R}^3$, then $\mathsf{tube}_r(\mathcal{M})$ is a filled segment of a cylinder of radius r whose axis is the line segment, with hemispheres at the two ends.



Definition 3. *For a d-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^D$, the **reach** of $\mathcal{M}$ is*

$$\mathsf{reach}(\mathcal{M}) \stackrel{\mathsf{def}}{=} \sup\{r > 0 : \mathsf{tube}_r(\mathcal{M}) \subseteq D(\mathcal{M})\}.$$

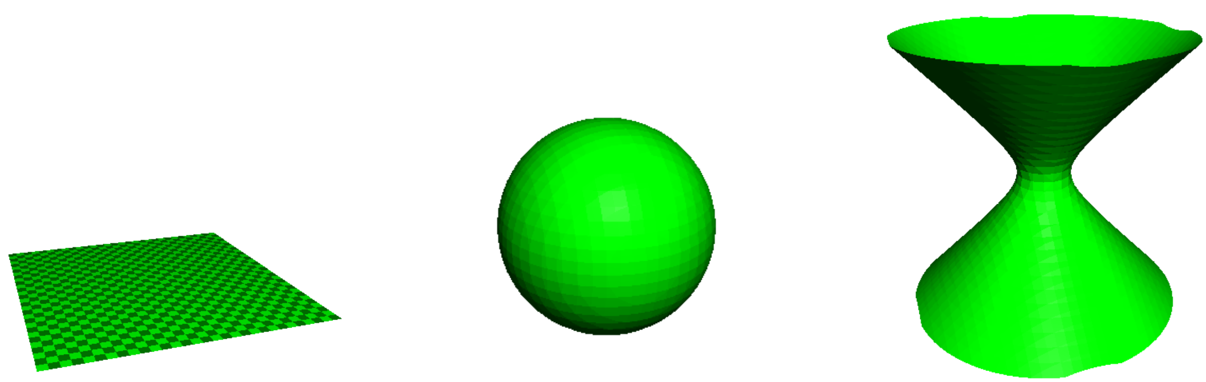
For example, if $\mathcal{M} = \partial\big(B_r(\mathbf{c})\big)$ for some $\mathbf{c} \in \mathbb{R}^D$, then

$$\mathsf{reach}(M) = r.$$

If $\mathcal{M}$ is an affine subspace, then

$$\mathsf{reach}(\mathcal{M}) = \infty.$$

The reach lets us bound many parameters of $\mathcal{M}$ (e.g. curvature, and self-avoidance). The larger the reach is, the "better" behaved the manifold is with respect to these features. As a demonstration, compare the three two-dimensional manifolds below, which have decreasing reach from left to right:



The reach also tells us for what portion of $\mathbb{R}^D$ (in terms of distance from $\mathcal{M}$) a projection onto $\mathcal{M}$ is well-defined.

Definition 4. *The **projection** onto $\mathcal{M}$ is the function $\Pi_{\mathcal{M}} : D(M) \to \overline{\mathcal{M}}$ defined by*

$$\Pi_{\mathcal{M}}(\mathbf{x}) := \mathbf{y} \in \overline{M} \textit{ closest to } \mathbf{x}.$$

Our goal is to prove a manifold embedding result for a general class of "nice" manifolds. Given $\mathcal{M} \subseteq \mathbb{R}^D$ that is compact (closed and bounded) with $\mathsf{reach}(\mathcal{M}) > 0$, we want to find $A \in \mathbb{R}^{m \times D}$ with $m << D$ such that

$$(1-\epsilon)\|\mathbf{x}-\mathbf{y}\|_2 \leq \|A(\mathbf{x}-\mathbf{y})\|_2 \leq (1+\epsilon)\|\mathbf{x}-\mathbf{y}\|_2 \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^D \tag{1}$$

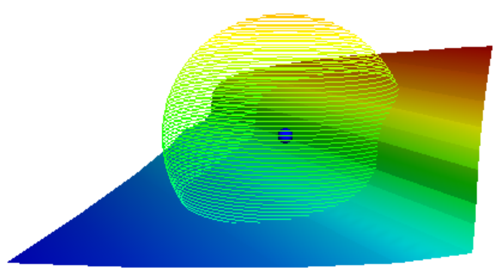
For simplicity, we will prove (1) up to precision $\delta \in \mathbb{R}^+$, a quantity we can make small inexpensively.

We will approach this result in the same way we proved the Johnson-Lindenstrauss subspace embedding result (Lemma 3, Lecture 14). Recall that to prove the subspace embedding result, we needed

1. covering numbers for unit balls, since they "encode the geometry" of a subspace. Here the manifold takes the place of the unit ball, and so we need covering numbers of manifolds; and then
2. to apply Johnson-Lindenstrauss to a minimal cover and do "a little work."

2 Covering Numbers for $\mathcal{M} \subseteq \mathbb{R}^D$

Here we describe the idea for bounding the covering number of compact manifolds with positive reach: We need to know "how much" of the d-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^D$ is contained in a (D-dimensional) ball centered at a point on the interior of the manifold. That is, given $\mathbf{x} \in \mathcal{M}^{\circ}$, what is the d-dimensional volume of $\mathcal{M} \cap B_r(\mathbf{x})$?



We will want to find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathsf{Vol}(B_r(\mathbf{p}) \cap \mathcal{M}) \geq f(r) \quad \text{for all } \mathbf{p} \in \mathcal{M},$$

as then we can bound the size of a minimal cover of $\mathcal{M}$ by

$$\begin{aligned} C_r(\mathcal{M}) &\leq P_r(\mathcal{M}) \\ &\leq \frac{\mathsf{Vol}(\mathcal{M})}{\inf_{\mathbf{p}\in\mathcal{M}} \mathsf{Vol}(B_r(\mathbf{p}) \cap \mathcal{M})} \\ &\leq \frac{\mathsf{Vol}(\mathcal{M})}{f(r)}, \end{aligned}$$

where the first inequality is the packing estimate obtained in Lemma 1 of Lecture 14. In the next lecture we will focus on finding such an f.

References

[1] Partha Niyogi, Stephen Smale, Shmuel Weinberger. Finding the Homology of Submanifolds with High Condence from Random Samples. *J. Discrete Comput. Geom.*, 39(1): 419–441, 2008.

[2] Mira Bernstein, Vin De Silva, John C. Langford, Joshua B. Tenenbaum. Graph Approximations to Geodesics on Embedded Manifolds. 2000.

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[4] Mark A. Iwen, Mauro Maggioni. Approximation of Points on Low-Dimensional Manifolds Via Random Linear Projections. *J. CoRR*, 1204.3337, 2012.

[5] Armin Eftekhari, Michael B. Wakin. New Analysis of Manifold Embeddings and Signal Recovery from Compressive Measurements. *J. CoRR* , 1306.4748, 2013.