MTH 995: Spec Top Numer Anly and Oper Res Spring 2015

Lecture John ellipsoid — Jan 22, 2015

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1 Overview

In the last lecture we covered singular value decomposition (SVD) and principal component analysis (PCA).

In this lecture we will cover John Ellipsoid.

2 Definition

Def. Convex set: *A set* $K \subseteq \mathbb{R}^D$ *is convex*
$\iff \forall \alpha \in [0,1],\ \forall \vec{x}, \vec{y} \in K :$
$\alpha\vec{x} + (1-\alpha)\vec{y} \in K$

3 Theorem

3.1 Theorem 1 (Fritz John)

Theorem 1. *Let* $K \subseteq \mathbb{R}^D$ *be a convex compact set and K is symmetric about the origin (K = -K)*
Then there exists an ellipsoid $E \subseteq \mathbb{R}^D such\, that : E \subseteq K \subseteq \sqrt{D}E$

Note: *worst case happens when K is a cube*

Proof: Assume WLOG that $spanK = \mathbb{R}^D$.
Since E has non empty volume, we have :
$E = AB$
where A is a symmetric positive definite matrix in $\mathbb{R}^{DxD}$ and B is the unit ball.
From singular value decomposition we can write A as:
$A = U\Sigma U^T$
$\Rightarrow AB = U\Sigma U^T B = U\Sigma B$ since U^T does not change the unit ball
$\Rightarrow Volume\, of\, E = detA = \prod_{i=1}^{D} \lambda_i$

Lemma 1. *The convex body K contains an ellipsoid of maximal volume*

Proof:
Let $L = \{A \in \mathbb{R}^{DxD} | AB \subseteq K\}$
We know the following:
L is bounded
L is closed (proof is homework)
$detA$ is continuous
$\Rightarrow$ There is a maximal ellipsoid. Q.E.D

Back to the proof of the theorem
Let E be a maximal volume ellipsoid in K
E = AB, where A is a symmetric positive definite matrix in $\mathbb{R}^{DxD}$ and B is the unit ball. Note that A is full rank.
$\Rightarrow B \subseteq A^{-1}K$ and B is a maximal volume ellipsoid in $A^{-1}K$
Consider a 2-dimensional projection of $A^{-1}K$
Let $\vec{p} \in A^{-1}K \backslash B$
$\vec{p} = (a, 0)$
How large can $||\vec{p}||$ be? By bounding a from above we will get the factor $\sqrt{D}$.
Let

$$A_{t,\lambda} = \begin{pmatrix} e^t & 0 & .. & 0 \\ 0 & e^{-\lambda t} & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & e^{-\lambda t} \end{pmatrix} \in \mathbb{R}^{DxD}$$

We want $Vol(A_{t,\lambda}) = e^{t(1-(D-1)\lambda)} > 1$ *and* $A_{t,\lambda}B \subseteq A^{-1}K$ *to get a contradiction with B is the maximal volume ellipsoid*
Suppose $a > \sqrt{D}$
$\frac{1}{D-1} > \lambda > \frac{1}{a^2-1}$
Let L_1 be the tangent line from $\vec{p}$ to the circle B
Let L_2 be the tangent line from $\vec{p}$ to the ellipse $A_{t,\lambda}B$
We want $A_{t,\lambda}B \subseteq A^{-1}K$
$\iff$ $slope\,of\,L_1 < slope\,of\,\,L_2$
$\iff$ $f(t) = -(a^2-1)e^{-2\lambda t} - e^{2t} + a^2 > 0$
Maximizing the function at $t^* = \frac{ln(\lambda(a^2-1))}{2(1+\lambda} > 0$ give us $f(t) > 0$
Therefore, we have:
$Vol(A_{t,\lambda}) = e^{t(1-(D-1)\lambda)} > 1$ by choice of λ
$A_{t,\lambda}B \subseteq A^{-1}K$ by $f(t) > 0$
$\Rightarrow$ B is not the maximal volume ellipsoid
$\Rightarrow$ contradiction
$\Rightarrow a < \sqrt{D}$
$\Rightarrow B \subseteq A^{-1}K \subseteq \sqrt{D}B$
$\Rightarrow E \subseteq K \subseteq \sqrt{D}E$
Q.E.D

3.2 Theorem 2 (Fritz John)

Theorem 2. *E is unique when K is compact*
$\sqrt{D}$ factor is sharp (eg: the cube)

3.3 Theorem 3

Theorem 3. *Let $K \subset \mathbb{R}^D$ be convex with $Vol(K) > 0$ (K may not be symmetric)*
Then there exits an ellipsoid E centered at $\vec{c} \in \mathbb{R}^D$ such that: $E \subseteq K \subseteq \vec{c} + D(E - \vec{c})$

3.4 Theorem 4

Theorem 4. *E is unique when K is compact*
D cannot be decreased in general

4 Find best fit ellipsoid

Problem: *Given $\{\vec{x_1}, ..., \vec{x_n}\} \subset \mathbb{R}^D$, fit an ellipsoid around them.*
Define ellipse as
$E_A := \left\{\vec{x} \in \mathbb{R}^D | \vec{x}^T A \vec{x} \leq 1\right\}$
where A is positive definite and symmetric
$A = U \begin{pmatrix} \lambda_1 & .. & 0 \\ .. & .. & .. \\ 0 & .. & \lambda_D \end{pmatrix} U^T$
Length of semi-axis of E_A:
$\frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq .. \leq \frac{1}{\lambda_D}$
$Homework: prove\, Vol(E_A) = \sqrt{detA^{-1}}$
To compute E_A, we find $Min_{A \in \mathbb{R}^{DxD}} - log(detA)$
i.e., $det(A^{-1})$
subject to two constraint:
1.$(x_j)^T A x_j \leq 1$
2.A is symmetric and positive definite

5 Homework

Homework 1 (section 2.1):
Let $K \subseteq \mathbb{R}^D$ be a convex compact set and K is symmetric about the origin (K = -K)
B be the unit ball

$L = \{A \in \mathbb{R}^{DxD} | AB \subseteq K\}$
Prove that L is closed
Homework 2 (section 3):
Let $E_A := \left\{\vec{x} \in \mathbb{R}^D | \vec{x}^T A \vec{x} \leq 1\right\}$
Prove : $Vol(E_A) = \sqrt{det A^{-1}}$