| MTH 995-003: Intro to CS and Big Data | Spring 2014 |
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| Lecture 16 — 02/27, 2014 | |
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# 1 Overview

In the last lecture we discussed the RIP property, null space property of order $k$. In this lecture, we recall the topic of null space, and basis pursuit for recovery of sparse vectors.

# 2 Main Contents

**Lemma 1.** *Given $S \subseteq [N]$ and $\vec{x}, \vec{z} \in \mathbb{C}^N$, we have*

$$\left\|(\vec{x}-\vec{z})_{\overline{S}}\right\|_1 \leq \|\vec{z}\|_1 - \|\vec{x}\|_1 + \|(\vec{x}-\vec{z})_S\|_1 + 2\left\|\vec{x}_{\overline{S}}\right\|_1. \tag{1}$$

*Proof:* Applying the triangle equality twice we get that

$$\|\vec{x}\|_1 = \left\|\vec{x}_{\overline{S}}\right\|_1 + \|\vec{x}_S\|_1 \leq \left\|\vec{x}_{\overline{S}}\right\|_1 + \|(\vec{x}-\vec{z})_S\|_1 + \|\vec{z}_S\|_1, \tag{2}$$

and

$$\left\|(\vec{x}-\vec{z})_{\overline{S}}\right\|_1 \leq \left\|\vec{x}_{\overline{S}}\right\|_1 + \left\|\vec{z}_{\overline{S}}\right\|_1. \tag{3}$$

Adding these two inequalities and rearranging gives the result. □

**Theorem 1.** *Let $\vec{x} \in \mathbb{C}^N$. Suppose that $\exists \rho \in (0,1)$ s.t. $A \in \mathbb{C}^{m \times N}$ has the null space property*

$$\|\vec{v}_S\|_1 \leq \rho \left\|\vec{v}_{\overline{S}}\right\|_1 \qquad (\$)$$

*$\forall \vec{v} \in \ker(A)$ and $\forall S \subset [N]$ with $|S| \leq k$. Then, any vector $\vec{z}^{\#} \in \mathbb{C}^N$ satisfying*

$$\left\|\vec{z}^{\#}\right\|_1 \text{ is minimal over all } \vec{z} \in \mathbb{C}^N \text{ with } A\vec{z} = A\vec{x} \qquad (BP)$$

*will approximate $\vec{x}$ near optimally in the sense that*

$$\left\|\vec{x} - \vec{z}^{\#}\right\|_1 \leq \frac{2(1+\rho)}{1-\rho} \cdot \left(\inf_{\vec{z} \in \mathbb{C}^N,\ \|\vec{z}\|_0 \leq k} \|\vec{x}-\vec{z}\|_1\right). \tag{4}$$

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**Review:** Some results from Lecture 15.

1. Lecture 15, theorem 1.
   Subgaussian random matrices have the RIP of order $k$ for $\varepsilon$. Thus, their RI constants are small.

2. Lecture 15, theorem 2.
   The RIP of order $k$ for $\varepsilon$ will also have the null space property ($)

3. Theorem 1(today) tells us that a matrix with ($) will work with Basis Pursuit (BP) for the purposes of compressive sensing.

4. Lecture 4 tells us that BP can be solved efficiently via linear programming

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*Proof of Theorem 1*: Let $S \subseteq [N]$ be s.t. $\|\vec{x}_S\|_1 = \inf_{\|\vec{z}\|_0 \leq k} \|(\vec{x} - \vec{z})\|_1$
Since $\vec{z}^{\#}$ (as defined above) satisfies $A\vec{z}^{\#} = A\vec{x}$, we have $\left(\vec{x} - \vec{z}^{\#}\right) \in \ker(A)$. Thus, by Lemma 1

$$\left\| \left(\vec{x} - \vec{z}^{\#}\right)_{\overline{S}} \right\|_1 \leq \|\vec{z}^{\#}\|_1 - \|\vec{x}\|_1 + \left\| \left(\vec{x} - \vec{z}^{\#}\right)_S \right\|_1 + 2 \cdot \|\vec{x}_{\overline{S}}\|_1 \tag{5}$$

Since $\left\|\vec{z}^{\#}\right\|_1$ is minimal over all vectors satisfying the constraint $A\vec{z}^{\#} = A\vec{x}$, it implies that:

$$\left\| \left(\vec{x} - \vec{z}^{\#}\right)_{\overline{S}} \right\|_1 \leq \left\| \left(\vec{x} - \vec{z}^{\#}\right)_S \right\|_1 + 2 \cdot \|\vec{x}_{\overline{S}}\|_1 \tag{6}$$

$$\leq \rho \cdot \left\| \left(\vec{x} - \vec{z}^{\#}\right)_{\overline{S}} \right\|_1 + 2 \cdot \|\vec{x}_{\overline{S}}\|_1 \qquad \text{by (\$)} \tag{7}$$

This implies that

$$\left\| \left(\vec{x} - \vec{z}^{\#}\right)_{\overline{S}} \right\|_1 \leq \frac{2}{1-\rho} \|\vec{x}_{\overline{S}}\|_1 \tag{8}$$

Thus,

$$\left\| \left(\vec{x} - \vec{z}^{\#}\right) \right\|_1 = \left\| \left(\vec{x} - \vec{z}^{\#}\right)_{\overline{S}} \right\|_1 + \left\| \left(\vec{x} - \vec{z}^{\#}\right)_S \right\|_1 \tag{9}$$

$$\leq (1+\rho) \left\| \left(\vec{x} - \vec{z}^{\#}\right)_{\overline{S}} \right\|_1 \qquad \text{by (\$)} \tag{10}$$

$$\leq \frac{2(1+\rho)}{1-\rho} \|\vec{x}_{\overline{S}}\|_1 \tag{11}$$

The theorem now follows from our choice of $S \subseteq [N]$ above. □

*A better result*

**Theorem 2** (Theorem 6.12 in [1])**.** *Suppose that $A \in \mathbb{C}^{m \times N}$ satisfies $\varepsilon_{2k}(A) < \frac{4}{\sqrt{41}}$. Then for any $\vec{x} \in \mathbb{C}^N$ and $y \in \mathbb{C}^m$ with $\|(A\vec{x} - \vec{y})\|_2 \leq \nu$, a solution $\vec{z}^{\#}$ of the modified BP problem:*

$$\min_{\vec{z} \in C^N} \|\vec{z}\|_1 s.t. \|(A\vec{z} - \vec{y})\|_2 \leq \nu \tag{12}$$

*will approximate the true solution $\vec{x}$ with errors:*

$$\left\|\left(\vec{x}-\vec{z}^{\#}\right)\right\|_1 \leq C \cdot \left( \inf_{\|\vec{z}\|_0 \leq k} \|(\vec{x}-\vec{z})\|_1 \right) + D\sqrt{k} \cdot \nu \tag{13}$$

$$\left\|\left(\vec{x}-\vec{z}^{\#}\right)\right\|_2 \leq \frac{C}{\sqrt{k}} \cdot \left( \inf_{\|\vec{z}\|_0 \leq k} \|(\vec{x}-\vec{z})\|_1 \right) + D\nu \tag{14}$$

*where $C, D$ are constants only only depend on $\varepsilon_{2k}(A)$.*

The following will be useful later.

**Lemma 2.** *Suppose that $A \in \mathbb{C}^{m \times N}$ has $k^{th}$ R.I. constant $\varepsilon_k(A) \in (0,1)$. Then,*

$$\|A\vec{x}\|_2 \leq \sqrt{1+\varepsilon_k(A)} \left[ \frac{\|x\|_1}{\sqrt{k}} + \|x\|_2 \right] \qquad \forall \vec{x} \in \mathbb{C}^N. \tag{15}$$

*As a result*

$$\sigma_1(A) \leq \sqrt{1+\varepsilon_k(A)} \left( \sqrt{\frac{N}{k}} + 1 \right). \tag{16}$$

*Proof*:

We have

$$\|A\vec{x}\|_2 = \left\| A \left( \vec{x}_{S_0} + \vec{x}_{S_1} + ... + \vec{x}_{S_{\lfloor \frac{N}{k} \rfloor}} \right) \right\|_2 \tag{17}$$

where $|\vec{x}_{j_1}| \geq |\vec{x}_{j_2}| \geq ... \geq |\vec{x}_{j_N}|$ and $S_l = \{j_{lk+1}, ..., j_{lk+k}\} \qquad \forall l = 0, 1, ..., \lfloor \frac{N}{k} \rfloor$.

Thus,

$$\|A\vec{x}\|_2 \leq \sum_{l=0}^{\lfloor \frac{N}{k} \rfloor} \|A\vec{x}_{S_l}\|_2 \quad \text{(the triangle inequality)} \tag{18}$$

$$\leq \sqrt{1+\varepsilon_k(A)} \left[ \sum_{l=0}^{\lfloor \frac{N}{k} \rfloor} \|\vec{x}_{S_l}\|_2 \right] \quad \text{(by definition of } \varepsilon_k(A)) \tag{19}$$

$$\leq \sqrt{1+\varepsilon_k(A)} \left[ \|\vec{x}_{S_0}\|_2 + \sum_{l=0}^{\lfloor \frac{N}{k} \rfloor} \frac{\|\vec{x}_{S_l}\|_1}{\sqrt{k}} \right] \quad \text{(by Lemma 2 from Lecture 15)} \tag{20}$$

$$\leq \sqrt{1+\varepsilon_k(A)} \left[ \|\vec{x}\|_2 + \frac{\|\vec{x}\|_1}{\sqrt{k}} \right]. \tag{21}$$

$\square$

We have seen in Lectures 14 and 15 that the Johnson-Lindenstrauss Lemma implies the Restricted Isometry Property (RIP). It also turns out that the RIP implies the J-L Lemma (i.e., they are "nearly equivalent" up to a loss in the parameters). We will use this result later as well.

**Theorem 3** (Theorem 9.36 in [1]). *Let $P \subseteq \mathbb{R}^N$ have $|P| = M$. Suppose that $A \in \mathbb{R}^{m \times N}$ has $\varepsilon_{2k}(A) \leq \eta/4$ for some $\eta, \delta \in (0,1)$ and $k \geq 16 \cdot \ln(4M/\delta)$. Let $\vec{\psi} \in \mathbb{R}^N$ have i.i.d. Bernoulli entries (taking on $+1/-1$ with prob $0.5$ each), then:*

$$(1-\eta)\,\|\vec{x}\|_2^2 \leq \left\| A \cdot Diag(\vec{\psi}) \cdot \vec{x} \right\|_2^2 \leq (1+\eta)\,\|\vec{x}\|_2^2 \tag{22}$$

$$\forall \vec{x} \in P \quad \textit{with prob.} \geq 1-\delta.$$

Note: $\mathrm{Diag}(\vec{\psi}) \in \mathbb{R}^{N \times N}$ is a diagonal matrix with the entries of $\vec{\psi}$ on its diagonal.

## References


[1] Simon Foucart and Holger Rauhut. A Mathematical Introduction to Compressive Sensing. Birkhauser Basel, 2013.