MTH 995-003: Intro to CS and Big Data	Spring 2014
Lecture 15 — Feb 25, 2014	
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Contents

1 The R.I.P. and Basis pursuit

Definition 1. *A matrix $A \in \mathbb{R}^{m\times N}$ satisfies the restricted isometry property of order $k \in [N]$ for $\epsilon \in (0,1)$ if*

$$(1-\epsilon)\|\ \vec{x}\|_2^2 \le \|A\vec{x}\|_2^2 \le (1+\epsilon)\|\vec{x}\|_2^2 \tag{1}$$

for all vectors $\vec{x} \in \mathbb{R}^N$ with $\|\vec{x}\|_0 \le k$ (at most k nonzero entries)

Theorem 1. *let $p, \epsilon \in (0,1)$and $A \in \mathbb{R}^{m\times N}$ with i.i.d mean 0, variance 1, subgaussian entries (with parameter c), and choose*

$$m \ge \frac{32c(16c+1)}{(\sqrt{1+\epsilon}-1)^2} \cdot k \cdot \ln\left(\frac{eN\left(1+\frac{16}{\sqrt{1+\epsilon}-1}\right)}{k\cdot\sqrt[k]{(1-p)/2}}\right).$$

Then $\frac{1}{\sqrt{m}}A$ will have RIP of order k for ϵ with probability at least p.

Proof: Do as Homework problem 1. □

Definition 2. *Let $\vec{x} \in \mathbb{R}^N$. We let the support of $\vec{x}$ be*

supp($\vec{x}$) = The set of nonzero coordinates in $\vec{x}$.

Example 1. *If $\vec{x} = \begin{pmatrix}100\\0\\100\\0\end{pmatrix}$, then supp($\vec{x}$) = $\{1,3\}$.*

Definition 3. *Let $S \subseteq \{1,2,...,N\} = [N]$, and $\vec{x} \in \mathbb{R}^N$. Then, we let $\vec{x}_S = \vec{x}$ with all entries not in S set to zero. That is,*

$$(x_S)_j = \begin{cases} 0, & \text{if } j \ne S \\ x_j, & \text{if } j \in S \end{cases}$$

Definition 4. *Similarly, if $A \in \mathbb{R}^{m\times N}$, then $A_S :=$ the sub-matrix of A consisting of the columns of A indexed in S.*

Example 2. $\vec{x} = \begin{pmatrix}100\\100\\100\\100\end{pmatrix}$ *and A is the 4×4 identity, $A = I_{4\times 4}$, and $S = \{2,4\}$,then* $\vec{x}_S = \begin{pmatrix}0\\100\\0\\100\end{pmatrix}$ *and* $A_S = \begin{pmatrix}0 & 0\\1 & 0\\0 & 0\\0 & 1\end{pmatrix}$.

Example 3. $\vec{x}_{supp(\vec{x})} = \vec{x} \quad \forall \vec{x} \in \mathbb{C}^N$

Definition 5. *Let $A \in \mathbb{C}^{m\times N}$. Then k^{th} restricted isometry constant of A, $\epsilon_k(A)$, is*

$$\epsilon_k(A) := \max_{S\subseteq[N],|S|\le k} \sigma_1\left(A_S^* A_S - I_{|S|\times|S|}\right) = \max_{S\subseteq[N],|S|\le k} \max\left\{\left|\sigma_1^2(A_S) - 1\right|, \left|\sigma_{|S|}^2(A_S) - 1\right|\right\}.$$

Note: If $A \in \mathbb{R}^{m\times N}$ has the RIP of order k for ϵ, then $\epsilon_k(A) \le \epsilon$. Theorem 1 above is simply a means of bounding the restricted isometry constants of certain random matrices.

Any matrix with small RI constants will be useful for sparse approximation, as we will begin to see next.

Lemma 1. *Let $\vec{u}, \vec{v} \in \mathbb{C}^N$ be complex vectors s.t. $\|\vec{u}\|_0 \le k$, $\|\vec{v}\|_0 \le t$. If $supp(\vec{u}) \cap supp(\vec{v}) = \emptyset$, then*

$$| < A\vec{u}, A\vec{v} > | \le \epsilon_{k+t}(A) \cdot \|\vec{u}\|_2 \cdot \|\vec{v}\|_2.$$

Proof: Let $S := supp(\vec{u}) \cup supp(\vec{v})$ so that $|S| = t + k$. Since $\vec{u}$ and $\vec{v}$ have disjoint supports, we have $< \vec{u}, \vec{v} >= 0$. Thus,

$$\begin{aligned}
&| < A\vec{u}, A\vec{v} > | = | < A_S\vec{u}_S, A_S\vec{v}_S > |\\
&= | < A_S\vec{u}_S, A_S\vec{v}_S > - < \vec{u}_S, \vec{v}_S > |\\
&= | < (A_S^* A_S - I)\vec{u}_S, \vec{v}_S > |\\
&\le \|(A_S^* A_S - I)\| \cdot \|\vec{u}_S\|_2 \cdot \|\vec{v}_S\|_2 \quad \text{(by Hölder's Inequality)}\\
&\le \epsilon_{k+t}(A) \cdot \|\vec{u}_S\|_2 \cdot \|\vec{v}_S\|_2.
\end{aligned}$$

□

Note: If $\vec{x} \in \mathbb{C}^N$, $[N] = \cup_{j=0}^{\lfloor \frac{N}{k} \rfloor} S_j$ then $\vec{x} = \sum_{j=0}^{\lfloor \frac{N}{k} \rfloor} \vec{x}_{S_j}$ where $S_j \cap S_l = \emptyset$ if $l \neq j$, and $|S_j| \le k$ for all $j = 0, 1, \ldots, \lfloor \frac{N}{k} \rfloor$.

Definition 6. *For any $\vec{x} \in \mathbb{C}^N$, we will let its coordinates be ordered by magnitude s.t.*

$$|x_{i_1}| \ge |x_{i_2}| \ge .... \ge |x_{i_N}|$$

and we will set $S_0 = \{i_1, i_2, ..., i_k\}$, and $S_j = \{i_{1+jk}, i_{2+jk}, ..., i_{(j+1)k}\}$ for all $j \in \left[\lfloor \frac{N}{k} \rfloor\right]$.

Lemma 2. *If $\vec{u} \in \mathbb{C}^N$ and $\vec{V} \in \mathbb{C}^N$ satisfy*

$$\|u\|_\infty := \max_{i\in[N]} |u_i| \le \min_{j\in[N]} |v_j| \tag{2}$$

then $\|\vec{u}\|_2 \le \frac{\|\vec{v}\|_1}{\sqrt{N}}$.

Proof: Homework 2.

Theorem 2. *Suppose that* $\epsilon_{2k}(A) < \frac{1}{2}$ *for some* $A \in \mathbb{C}^{m\times N}$*.Then*

$$\|\vec{v}_S\|_1 \leq \frac{\epsilon_{2k}(A)}{1-2\epsilon_{2k}(A)}\|\vec{v}_{\overline{S}}\|_1 \tag{3}$$

for all $\vec{v} \in \ker(A)$*and* $\forall S \subseteq \{1,2,3,....,N\}$ *with* $|S| \leq k$*.*

Note: (3) holding as above is commonly referred to as the "null space property" with constant $\frac{\epsilon_{2k}(A)}{1-2\epsilon_{2k}(A)}$. If $A\vec{v} = 0$ then $\vec{v}$ cannot be k-sparse.

Proof: Let $S \subset [N]$ with $|S| \leq k$. If (3) holds for just S_0 w.r.t. at given $\vec{v}$ it is enough since

$$\|\vec{v}_S\|_1 \leq \|\vec{v}_{S_0}\|_1 \leq \frac{\epsilon_{2k}(A)}{1-2\epsilon_{2k}(A)}\|\vec{v}_{\overline{S_0}}\|_2 \leq \frac{\epsilon_{2k}(A)}{1-2\epsilon_{2k}(A)}\|\vec{v}_{\overline{S}}\|_2 \tag{4}$$

Therefore, it suffices to establish (3) for S_0 w.r.t. each $\vec{v} \in \ker(A)$

By definition of the Restricted Isometry (RI) constant, $\epsilon_{2k}(A) \geq \epsilon_k(A)$. Thus

$$(1-\epsilon_{2k}(A))\|\vec{v}_{S_0}\|_2^2 \leq (1-\epsilon_k(A))\|\vec{v}_{S_0}\|_2^2 \leq \|A\vec{v}_{S_0}\|_2^2, \tag{5}$$

and so

$$\|v\vec{S}0\|_2^2 \leq \frac{1}{1-\epsilon_{2k}(A)}\|A\vec{v}_{S_0}\|_2^2 = \frac{1}{1-\epsilon_{2k}(A)}\langle A\vec{v}_{S_0}, A\vec{v}_{S_0}\rangle.$$

Since $\vec{v} \in \ker(A)$, we have that $A\vec{v}_{S_0} = -\sum_{j=1}^{\lfloor \frac{N}{k} \rfloor} Av_{S_j}$. Thus, Lemma 1 implies that

$$\|\vec{v}_{S_0}\|_2^2 \leq \frac{1}{1-\epsilon_{2k}(A)}\sum_{j\geq 1}\langle A\vec{v}_{S_0}, -A\vec{v}_{S_j}\rangle \leq \frac{\epsilon_{2k}(A)}{1-\epsilon_{2k}(A)}\sum_{j\geq 1}\|\vec{v}_{S_0}\|_2.\|\vec{v}_{S_j}\|_2$$

and so we get that

$$\|\vec{v}_{S_0}\| \leq \frac{\epsilon_{2k}(A)}{1-\epsilon_{2k}(A)}\sum_{j=1}^{\lfloor \frac{N}{k} \rfloor}\|\vec{v}_{S_j}\|_2. \tag{6}$$

Now, Lemma 2 tells us that $\|\vec{v}_{S_j}\|_2 \leq \frac{\|\vec{v}_{S_{j-1}}\|_1}{\sqrt{k}}, \quad \forall j \geq 1$. Thus,

$$\|\vec{v}_{S_0}\|_2 \leq \frac{1}{\sqrt{k}} \cdot \frac{\epsilon_{2k}(A)}{1-\epsilon_{2k}(A)} \cdot \|\vec{v}\|_1.$$

Now to finish, we note that Hölder's Inequality implies that $\|\vec{v}_{S_0}\|_1 \leq \|\vec{v}_{S_0}\|_2 \cdot \sqrt{k}$. Therefore, a rearrangement of the last inequality will give us the desired result. □

1.1 Homework Problems:- due March 21^{st} (Tue.)

Homework 1: Prove theorem 1.
Homework 2: Prove lemma 2.

References

[1] Simon Foucart and Holger Rauhut. A Mathematical Introduction to Compressive Sensing. Birkhauser Basel, 2013.