MTH 995-001: Intro to CS and Big Data	Spring 2015

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1 Overview

In the last lecture we talked about:

- Proving the Rademacher Chaos Lemma

In this lecture we will:

- Show and Prove that RIP implies the Johnson-Lindenstrauss Lemma

2 RIP implies the Johnson-Lindenstrauss Lemma

Theorem 1 (RIP implies the Johnson-Lindenstrauss Lemma)**.** *Let $P \subseteq \mathbb{R}^N$ have $|P| = M$. Choose $\delta, \eta \in (0,1)$ and suppose that $A \in \mathbb{R}^{m \times N}$ has $\epsilon_{2k}(A) \leq \frac{\eta}{4}$ Let $\vec{\psi} \in \mathbb{R}^N$ have i.i.d. Bernoulli entries (± 1 with probability $\frac{1}{2}$). Suppose, $k \geq 16 \ln\left(\frac{4M}{\delta}\right)$. Then,*

$$(1-\eta)\|\vec{x}\|_2^2 \leq \left\|A\,\mathrm{diag}\left(\vec{\psi}\right)\vec{x}\right\|_2^2 \leq (1+\eta)\|\vec{x}\|_2^2$$

$\forall \vec{x} \in P$ with probability $\geq 1-\delta$.

Proof: Let $\vec{x} \in P$ and assume that $\|\vec{x}\|_2 = 1$.

- We partition $[N]$ into disjoint sets $S_1, \ldots, S_{\lceil \frac{N}{k} \rceil} \subseteq [N]$ (i.e. a partition of $[N]$), each of size k. That is

  $$|S_1| = |S_2| = \cdots = \left|S_{\lceil \frac{N}{k} \rceil}\right| = k$$

  Where $S_1 \subseteq [N]$ contains the indexes of the k-largest magnitude entries of $\vec{x}$, $S_2 \subseteq [N]$ contains the indexes of the k second-largest magnitude entries of $\vec{x}$, etc.

- $D_{\vec{\psi}} = \mathrm{diag}\left(\vec{\psi}\right)$, that is, a diagonal matrix with $\vec{\psi} \in \mathbb{R}^N$ on the diagonal

- $$\begin{aligned}\left\|AD_{\vec{\psi}}\vec{x}\right\|_2^2 &= \left\langle AD_{\vec{\psi}}\left(\vec{x}_{S_1} + \vec{x}_{\overline{S_1}}\right), AD_{\vec{\psi}}\left(\vec{x}_{S_1} + \vec{x}_{\overline{S_1}}\right)\right\rangle \\ &= \left\|AD_{\vec{\psi}}\vec{x}_{S_1}\right\|_2^2 + 2\left\langle AD_{\vec{\psi}}\vec{x}_{S_1}, AD_{\vec{\psi}}\vec{x}_{\overline{S_1}}\right\rangle + \left\|AD_{\vec{\psi}}\vec{x}_{\overline{S_1}}\right\|_2^2\end{aligned}$$

- Expand the last term and rearrange to get

$$\left\|AD_{\vec{\psi}}\vec{x}\right\|_2^2 = \underbrace{\sum_{j=1}^{\lceil \frac{N}{k} \rceil} \left\|AD_{\vec{\psi}}\vec{x}_{S_j}\right\|_2^2}_{\text{term 1}} + \underbrace{2\left\langle AD_{\vec{\psi}}\vec{x}_{S_1}, AD_{\vec{\psi}}\vec{x}_{\overline{S_1}}\right\rangle}_{\text{term 2}} + \underbrace{\sum_{\substack{i\neq j\\ j,i\geq 2}}^{\lceil \frac{N}{k} \rceil} \left\langle AD_{\vec{\psi}}\vec{x}_{S_j}, AD_{\vec{\psi}}\vec{x}_{S_i}\right\rangle}_{\text{term 3}}$$

Bound on Term 1. *Note*: $\epsilon_k(A) \leq \epsilon_{2k}(A) \leq \frac{\eta}{4}$

$$\begin{aligned}
\left(1-\frac{\eta}{4}\right)\|\vec{x}\|_2^2 &= \left(1-\frac{\eta}{4}\right)\left\|D_{\vec{\psi}}\vec{x}\right\|_2^2 && D_{\vec{\psi}} \text{ is unitary}\\
&\leq \left(1-\frac{\eta}{4}\right)\sum_{j=1}^{\lceil \frac{N}{k} \rceil}\left\|D_{\vec{\psi}}\vec{x}_{S_j}\right\|_2^2\\
&\leq \sum_{j=1}^{\lceil \frac{N}{k} \rceil}\left\|AD_{\vec{\psi}}\vec{x}_{S_j}\right\|_2^2 && \text{By RIP}\\
&\leq \left(1+\frac{\eta}{4}\right)\sum_{j=1}^{\lceil \frac{N}{k} \rceil}\left\|D_{\vec{\psi}}\vec{x}_{S_j}\right\|_2^2\\
&= \left(1+\frac{\eta}{4}\right)\|\vec{x}\|_2^2
\end{aligned}$$

In specific,

$$\left(1-\frac{\eta}{4}\right)\|\vec{x}\|_2^2 \leq \sum_{j=1}^{\lceil \frac{N}{k} \rceil}\left\|AD_{\vec{\psi}}\vec{x}_{S_j}\right\|_2^2 \leq \left(1+\frac{\eta}{4}\right)\|\vec{x}\|_2^2$$

Bound on Term 2.

$$\begin{aligned}
X &:= \langle AD_{\vec{\psi}}\vec{x}_{S_1}, AD_{\vec{\psi}}\vec{x}_{\overline{S_1}}\rangle\\
&= \langle A_{S_1}D_{\vec{\psi}_{S_1}}\vec{x}_{S_1}, A_{\overline{S_1}}D_{\vec{\psi}_{\overline{S_1}}}\vec{x}_{\overline{S_1}}\rangle\\
&= \langle \underbrace{D_{\vec{x}_{\overline{S_1}}}A^*_{\overline{S_1}}A_{S_1}D_{\vec{\psi}_{S_1}}\vec{x}_{S_1}}_{\vec{a}}, \vec{x}_{\overline{S_1}}\rangle && \text{Note } \vec{a} \text{ and } \vec{x}_{\overline{S_1}} \text{ are independent}
\end{aligned}$$

- We can use Hoeffding's inequality from lecture 11a to bound the inner product because of the independence.
- We need a bound for $\vec{a}$.

$$
\begin{aligned}
\left\| D_{\vec{x}_{\overline{S_1}}} A^*_{\overline{S_1}} A_{S_1} D_{\vec{\psi}_{S_1}} \vec{x}_{S_1} \right\|_2 &= \sup_{\|\vec{z}\|=1} \sum_{j\geq 2}^{\lceil \frac{N}{k} \rceil} \langle \vec{z}_{S_j}, D_{\vec{x}_{\overline{S_1}}} A^*_{\overline{S_1}} A_{S_1} D_{\vec{\psi}_{S_1}} \vec{x}_{S_1} \rangle \\
&= \sup_{\|\vec{z}\|=1} \sum_{j\geq 2}^{\lceil \frac{N}{k} \rceil} \langle \vec{z}_{S_j}, D_{\vec{x}_{S_j}} A^*_{S_j} A_{S_1} D_{\vec{\psi}_{S_1}} \vec{x}_{S_1} \rangle \\
&\leq \sup_{\|\vec{z}\|=1} \sum_{j\geq 2}^{\lceil \frac{N}{k} \rceil} \left\| \vec{z}_{S_j} \right\|_2 \left\| D_{\vec{x}_{S_j}} A^*_{S_j} A_{S_1} D_{\vec{\psi}_{S_1}} \right\|_2 \left\| \vec{x}_{S_1} \right\|_2 && \text{Cauchy-Schwarz} \\
&\leq \sup_{\|\vec{z}\|=1} \sum_{j\geq 2}^{\lceil \frac{N}{k} \rceil} \left\| \vec{z}_{S_j} \right\|_2 \overbrace{\left\| \vec{x}_{S_1} \right\|_2}^{\text{bounded by 1}} \left\| \vec{x}_{S_j} \right\|_\infty \left\| A^*_{S_j} A_{S_1} \right\|_2 \\
&\leq \sup_{\|\vec{z}\|=1} \sum_{j\geq 2}^{\lceil \frac{N}{k} \rceil} \left\| \vec{z}_{S_j} \right\|_2 \frac{\left\| \vec{x}_{S_{j-1}} \right\|_\infty}{\sqrt{k}} \epsilon_{2k}(A) && \text{by Lemma 1 Lect. 15} \\
&\leq \frac{\epsilon_{2k}(A)}{\sqrt{k}} \sup_{\|\vec{z}\|=1} \sum_{j\geq 2}^{\lceil \frac{N}{k} \rceil} \left\| \vec{x}_{S_{j-1}} \right\|_2 \left\| \vec{z}_{S_j} \right\|_2 \\
&\leq \frac{\epsilon_{2k}(A)}{\sqrt{k}} \sup_{\|\vec{z}\|=1} \sum_{j\geq 2}^{\lceil \frac{N}{k} \rceil} \frac{\overbrace{\left\| \vec{x}_{S_{j-1}} \right\|_2^2}^{\text{bounded by 1}} \overbrace{\left\| \vec{z}_{S_j} \right\|_2^2}^{\text{bounded by 1}}}{2} && \text{Cauchy-Schwarz} \\
&\leq \frac{\epsilon_{2k}(A)}{\sqrt{k}} \quad \left(\leq \frac{\eta}{4} \right)
\end{aligned}
$$

Applying Hoeffding's inequality from lecture 11a yields,

$$
\begin{aligned}
\mathbb{P}\left(\left| 2 \left\langle AD_{\vec{\psi}} \vec{x}_{S_1}, AD_{\vec{\psi}} \vec{x}_{\overline{S_1}} \right\rangle \right| \geq t \right) &\leq 2 \exp\left(-\frac{t^2 k}{2\epsilon_{2k}^2(A)} \right) \\
&\leq 2 \exp\left(-\frac{8kt^2}{\eta^2} \right) && \forall t > 0
\end{aligned}
$$

Bound on Term 3. Term 3 can be rewritten as

$$
\sum_{\substack{i\neq j \\ j,i\geq 2}}^{\lceil \frac{N}{k} \rceil} \langle \vec{\psi}, D_{\vec{x}_{S_j}} \left(A^*_{S_j} A_{S_i} \right) D_{\vec{x}_{S_i}} \vec{\psi} \rangle = \vec{\psi}^* B \vec{\psi}
$$

where

$$
B_{k,l} = \begin{cases} x_k \vec{a}_k^* \vec{a}_l x_l & k,l \in \overline{S_1} \text{ and belong to different } S_j \\ 0 & \text{otherwise} \end{cases}
$$

Note:

- B has 0's on the diagonal
- B is symmetric

We can use Lemma 2 (Rademacher Chaos) from Lecture 22 to bound term 3. But first we have to bound $\|B\|_{\mathrm{F}}$ and $\|B\|_2$. A bound on $\|B\|_2$ can be found using a method similar to that in Lecture 23.

$$\begin{aligned}
\|B\|_{\mathrm{F}}^2 &= \sum_{\substack{j,k\geq 2\\ j\neq k}} \sum_{i\in S_j} \sum_{j\in S_k} \left(x_i \vec{a}_i^* \vec{a}_l x_l\right)^2 \\
&= \sum_{\substack{j,k\geq 2\\ j\neq k}} \sum_{i\in S_j} x_i^2 \left\| D_{\vec{x}_{S_k}} A_{S_k}^* a_i \right\|_2^2 \\
&\leq \sum_{\substack{j,k\geq 2\\ j\neq k}} \sum_{i\in S_j} x_i^2 \left\| \vec{x}_{S_k} \right\|_\infty^2 \left\| A_{S_k}^* a_i \right\|_2^2 \\
&\leq \left(\epsilon_{2k}\left(A\right)\right)^2 \sum_{\substack{j,k\geq 2\\ j\neq k}} \left\| \vec{x}_{S_j} \right\|_2^2 \frac{\left\| \vec{x}_{S_{k-1}} \right\|_2^2}{k} && \text{by HW problems below}
\end{aligned}$$

Thus,

$$\sum_{\substack{i\neq j\\ j,i\geq 2}}^{\lceil \frac{N}{k} \rceil} \left\langle AD_{\vec{\psi}} \vec{x}_{S_j}, AD_{\vec{\psi}} \vec{x}_{S_i} \right\rangle \leq \frac{\left(\epsilon_{2k}\left(A\right)\right)^2}{k}$$

Combining the Bounds. For term 2 choose $t = \frac{\eta}{8}$. Similarly for term 3 choose t as a function of η. Then combine to get the statement in Theorem 1. $\square$

3 Homework

Problem 1 Prove

$$\left\| \vec{x}_{S_j} \right\|_\infty \leq \frac{\left\| \vec{x}_{S_{j-1}} \right\|_2}{\sqrt{k}}$$

Problem 2 Prove

$$\left\| A_{S_j}^* \vec{a}_i \right\|_2^2 \leq \left(\epsilon_{2k}\left(A\right)\right)^2 \forall i \in S_j$$