MTH 995-001: Intro to CS and Big Data	Spring 2015
Lecture 23 — April 9, 2019	
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1 Overview

In the last lecture we talked about:

- Decoupling
- Rademacher Chaos Lemma (Statement only, no Proof)

In this lecture we will:

- Give a Proof of the Rademacher Chaos Lemma

2 Proof of Rademacher Chaos Lemma

The following is a proof of Rademacher Chaos stated in Lemma 2 of Lecture 22. Recall that B is symmetric & has 0's on the diagonal, $\vec{\psi}$ is Bernoulli, and $\vec{\psi}'$ is an independent copy of $\vec{\psi}$.

Proof: Decoupling (Lemma 1 in Lecture 22) gives us

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\theta\vec{\psi}^* B\vec{\psi}\right)\right] &\leq \mathbb{E}\left[\exp\left(4\theta\vec{\psi}^* B\vec{\psi}'\right)\right] \\
&= \mathop{\mathbb{E}}_{\vec{\psi}}\mathop{\mathbb{E}}_{\vec{\psi}'}\left[\exp\left(4\theta\sum_{k=1}^{N}\vec{\psi}'\left(\vec{\psi}_j B_{jk}\right)\right)\right] && \text{By Lemma 2, Lect. 13} \\
&\leq E_{\vec{\psi}}\left[\exp\left(\frac{(4\theta)^2}{2}\sum_{k=1}^{N}\left(\sum_{j=1}^{N}\vec{\psi}_j B_{jk}\right)^2\right)\right] && \text{By Def. of Subgaussian RV Params}
\end{aligned}$$

Aside:

$$\sum_{k=1}^{N}\left(\sum_{j=1}^{N}\vec{\psi}_j B_{jk}\right)^2 = \|B^*\vec{\psi}\|^2 \qquad B\text{'s symmetry} \implies B_{jk} = (B^*)_{kj}$$
$$= \langle B^*\vec{\psi}, B^*\vec{\psi}\rangle$$
$$= \langle \vec{\psi}, BB^*\vec{\psi}\rangle$$
$$= \vec{\psi}^* B^2 \vec{\psi}$$

Thus we now have,

$$\mathbb{E}\left[\exp\left(\theta\vec{\psi}^* B\vec{\psi}\right)\right] \leq \mathbb{E}\left[\exp\left(8\theta^2\vec{\psi}^* B^2\vec{\psi}\right)\right] \tag{1}$$

Then we can estimate the RHS of Equation 1.

$$\mathbb{E}\left[\exp\left(8\theta^2\vec{\psi}^* B^2\vec{\psi}\right)\right] = \exp\left[8\theta^2\,\mathrm{tr}(B^2)\right]\mathbb{E}\left[\exp\left(8\theta^2\left(\vec{\psi}^*\left(B^2 - \mathrm{diag}(B^2)\right)\vec{\psi}\right)\right)\right]$$

We then repeat the previous argument to get:

$$\mathbb{E}\left[\exp\left(8\theta^2\vec{\psi}^* B\vec{\psi}\right)\right] \leq \exp\left[8\theta^2\|B\|_{\mathrm{F}}^2\right]\mathbb{E}\left[\exp\left(32\theta^2\vec{\psi}^* B^2\vec{\psi}\right)\right] \qquad \text{By Decoupling Lemma}$$
$$\leq \exp\left[8\theta^2\|B\|_{\mathrm{F}}^2\right]\mathbb{E}\left[\exp\left(512\theta^4\vec{\psi}^* B^4\vec{\psi}\right)\right] \qquad \text{Lecture 22 Subgaussian argument}$$

Aside:

$$\vec{\psi}^* B^4\vec{\psi} = \left(|B|\,\vec{\psi}\right)^* B^2\left(|B|\,\vec{\psi}\right)$$
$$\leq \|B\|^2\,\vec{\psi}^* B^2\vec{\psi}$$

where $|B| = V\begin{pmatrix}|\lambda_1| & & \\ & \ddots & \\ & & |\lambda_N|\end{pmatrix}V^*$ and $B = V\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}V^*$ with λ_i as the eigenvalues and V are eigenvectors.

Therefore, Equation 1 yields

$$\mathbb{E}\left[\exp\left(8\theta^2\vec{\psi}^* B\vec{\psi}\right)\right] \leq \exp\left[8\theta^2\|B\|_{\mathrm{F}}^2\right]\mathbb{E}\left[\exp\left(512\theta^4\|B\|_{\mathrm{op}}^2\,\vec{\psi}^* B^2\vec{\psi}\right)\right]$$
$$= \exp\left[8\theta^2\|B\|_{\mathrm{F}}^2\right]\mathbb{E}\left[\left(\exp\left(8\theta^2\vec{\psi}^* B^2\vec{\psi}\right)\right)^{64\theta^2\|B\|_{\mathrm{op}}^2}\right]$$

If θ is chosen such that $64\theta^2 \|B\|_{\text{op}}^2 < 1$, then Jensen's Inequality gives us

$$\mathbb{E}\left[\exp\left(8\theta^2 \vec{\psi}^* B \vec{\psi}\right)\right] \leq \exp\left[8\theta^2 \|B\|_{\text{F}}^2\right] \mathbb{E}\left[\exp\left(8\theta^2 \vec{\psi}^* B^2 \vec{\psi}\right)\right]^{64\theta^2 \|B\|_{\text{op}}^2}$$

Which can be rearranged with algebra to get

$$\mathbb{E}\left[\exp\left(8\theta^2 \vec{\psi}^* B \vec{\psi}\right)\right] \leq \exp\left(\frac{8\theta^2 \|B\|_{\text{F}}^2}{1 - 64\theta^2 \|B\|_{\text{op}}^2}\right) \qquad \text{when } 64\theta^2 \|B\|_{\text{op}}^2 < 1$$

By Equation 1, we now see

$$\mathbb{E}\left[\exp\left(\theta \vec{\psi}^* B \vec{\psi}\right)\right] \leq \exp\left(\frac{8\theta^2 \|B\|_{\text{F}}^2}{1 - 64\theta^2 \|B\|_{\text{op}}^2}\right)$$

Thus,

$$\begin{aligned}
\mathbb{P}\left[\vec{\psi}^* B \vec{\psi} \geq t\right] &= \mathbb{P}\left[\exp\left(\theta \vec{\psi}^* B \vec{\psi}\right) \geq \exp(\theta t)\right] \\
&\leq \exp(-\theta t)\, \mathbb{E}\left[\exp\left(\theta \vec{\psi}^* B \vec{\psi}\right)\right] \\
&\leq \exp\left(-\theta t + \frac{8\theta^2 \|B\|_{\text{F}}^2}{1 - 64\theta^2 \|B\|_{\text{op}}^2}\right) \\
&\leq \exp\left(-\theta t + \frac{32\theta^2 \|B\|_{\text{F}}^2}{3}\right) \qquad \text{using } \theta \leq \tfrac{1}{16\|B\|_{\text{op}}}
\end{aligned}$$

To get the statement in the proof we optimize over θ subject to $\theta < \frac{1}{16\|B\|_{\text{op}}}$. This only gives us half of the statement (as the original statement had $\mathbb{P}\left[\left|\vec{\psi}^* B \vec{\psi}\right| \geq t\right]$) so we repeat the argument for $-B$. □