Any of the following exercises are fair game for appearing in a slightly altered state on an exam. I suggest that you write up your solutions **neatly** in your own handwriting, or .tex them up, so that you can easily use them for studying later. All solutions should be your own! Feel free to discuss solutions with others, but don't write anything down that you don't understand (and write it all yourself!). **Only a subset of the problems will be graded, so do them all to the best of your ability, and be aware: even if there is only one problem that you don't do well, it might be one that's graded! So, try to do them all well!**

1. Do problem 1 on page 61 of Folland. For part (c) you should look up and apply the Dominated Convergence Theorem in order to make your solution rigorous. **This problem shows that a function's Fourier series may not converge uniformly to the function on $[-\pi, \pi]$ even if it is 2π periodic, piecewise smooth, and has only one discontinuity!** The upshot: continuity is both a necessary and sufficient condition for the uniform convergence of an arbitrary 2π periodic and piecewise smooth function's Fourier series to the function everywhere on $[-\pi, \pi]$ (see Theorem 2.5 on page 41 of Folland for sufficiency).

2. Recall the we defined the class $\mathcal{C}^k$ of functions $f : [-\pi, \pi] \to \mathbb{C}$ to be those with a Riemann integrable k^{th} derivative on $[-\pi, \pi]$. We are also assuming periodicity so that all lesser derivatives of f, $f^{(l)}$, are continuous with $f^{(l)}(\pi) = f^{(l)}(-\pi)$ for all $l = 0, \dots, k-1$. For any $f \in \mathcal{C}^k$ we then showed that:

   (a) **THM 2/8 (A):** $c_n = c_n^{(k)}/(in)^k$ for all $n \neq 0$ (where $c_n^{(l)}$ is the n^{th} Fourier series coefficient of $f^{(l)}$, with $c_n = c_n^{(0)}$).

   (b) **THM 2/8 (B):** If $f \in \mathcal{C}^k$ with $k \geq 2$, then $f(\theta) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \mathbb{e}^{\mathbb{i}n\theta}$ for all $\theta \in [-\pi, \pi]$ (uniformly).

   Use fact (a) to strengthen fact (b) via the proof idea of Theorem 2.5 in Folland (page 41). That is, use the Cauchy-Schwarz inequality to prove that, for every $f \in \mathcal{C}^k$ with $k \geq 1$, there exists a constant a $B \in \mathbb{R}^+$ (which only depends on $f^{(k)}$ and k) such that

   $$\left| f(\theta) - \sum_{n=-N}^{N} c_n \mathbb{e}^{\mathbb{i}n\theta} \right| \leq \frac{B}{N^{k-\frac{1}{2}}} \tag{1}$$

   holds for all $N \in \mathbb{Z}^+$ and $\theta \in [-\pi, \pi]$. This proves that convergence of the Fourier series is both uniform as soon as f is smooth enough, and also generally faster the smoother f is.

3. The theorem you proved for #2 is useful in applications. Let's try to understand it a little better with an example: Consider the function $f(x) = \cos(100x)$. Note that $f \in \mathcal{C}^k$ for any $k \in \mathbb{Z}^+$ you like.

   (a) Look back at your proof of Theorem 2 and figure out how large B is for $f(x) = \cos(100x)$ when $k = 13$. How does this B vary with k in general?

   (b) How large does the error bound in equation 1 tell you have to take N before you can be sure that your error will always be less than .001 for all $\omega \in [-\pi, \pi]$ when using $k = 13$ for $f(x) = \cos(100x)$?

   (c) How large will your actual error be once you pick any $N \geq 100$ for $f(x) = \cos(100x)$?

   (d) How large does your error bound in equation 1 tell you have to take N before you can be sure that your error will always be less than 10^{-16} for all $\omega \in [-\pi, \pi]$ as you let $k \to \infty$ for $f(x) = \cos(100x)$?

4. Suppose you know that a function $f : [-\pi, \pi] \to \mathbb{C}$ you want to learn about is composed of exactly one frequency component. That is, that $f(x) = A\mathbb{e}^{\mathbb{i}\omega x}$ for unknown parameters $A \in \mathbb{C}$ and $\omega \in \mathbb{Z} \cap [-127627, 127627]$. Use **THM 2/13** from class (see below) together with the Chinese Remainder Theorem in order to show that you can learn both A and ω by sampling f at just 51 different points $\in [-\pi, \pi]$.

   (a) **THM 2/13:** Let $\tilde{c}_n := \frac{(-1)^n}{N} \sum_{k=0}^{N-1} f\left(-\pi + k \cdot \frac{2\pi}{N}\right) \mathbb{e}^{\frac{-2\pi \mathbb{i} n k}{N}}$. Then,

   $$\tilde{c}_n = \sum_{q=-\infty}^{\infty} (-1)^q \, c_{n+Nq} = \sum_{m \equiv n \bmod N} (-1)^{\frac{m-n}{N}} c_m,$$

   where c_n is the n^{th} Fourier series coefficient of $f : [-\pi, \pi] \to \mathbb{C}$.

   HINT: You will want to use 6 sets of equally spaced samples in $[-\pi, \pi]$, each associated with a different prime number.

5. Do problems 6 and 7 on page 68 of Folland.

6. Do problems 8 and 9 on page 68 of Folland.

7. Do problems 1 and 2 on page 71 of Folland.

8. Do problem 3 on page 71 of Folland.

9. Do problem 4 on page 71 of Folland.

10. Do problems 6 and 7 on page 71 of Folland.