Do any 7 of the following 10 exercises of your choice. Write up your solutions neatly in your own handwriting! Most problems are from the book, which is available online here: `http://www-elec.inaoep.mx/~rogerio/FourierAnalysisUno.pdf`.

1. Prove the following facts about the Dirichlet kernel,

$$D_N(y) := \frac{1}{2\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)y\right)}{\sin\left(\frac{y}{2}\right)} = \frac{1}{2\pi}\sum_{k=-N}^{N} \mathbb{e}^{\mathbb{i}ky},$$

that we discussed in class. As usual, you should show your work.

   (a) Prove that $D_N(0) = \frac{2N+1}{2\pi}$.

   (b) Prove that

$$D_N(\pi) = \begin{cases} \frac{1}{2\pi} & \text{if } N \text{ is even} \\ -\frac{1}{2\pi} & \text{if } N \text{ is odd} \end{cases}.$$

   (c) Prove that the Dirichlet has kernel has exactly $2N$ zeros in $[-\pi, \pi]$. What are they? (Derive an equation for them.)

2. A function $f : [-\pi, \pi] \to \mathbb{C}$ can always be split into its imaginary and real parts, $f_1 : [-\pi, \pi] \to \mathbb{R}$ and $f_2 : [-\pi, \pi] \to \mathbb{R}$, such that

$$f(x) = f_1(x) + \mathbb{i}f_2(x)$$

holds for all $x \in [-\pi, \pi]$. Prove the following facts about $f : [-\pi, \pi] \to \mathbb{C}$.

   (a) Prove that the Fourier series coefficients, c_n, of $f : [-\pi, \pi] \to \mathbb{C}$ always satisfy $c_n = \tilde{c}_n + \mathbb{i}\dot{c}_n$, where $\tilde{c}_n$ and $\dot{c}_n$ denote the Fourier series coefficients of f_1 and f_2, respectively. This should be easy (i.e., don't think too hard).

   (b) We will say that a function $f : [-\pi, \pi] \to \mathbb{C}$ is Riemann integrable if both its real and imaginary parts, $f_1 : [-\pi, \pi] \to \mathbb{R}$ and $f_2 : [-\pi, \pi] \to \mathbb{R}$, are Riemann integrable. Prove that the real-valued function $|f(x)|^2$ is Riemann integrable on $[-\pi, \pi]$ whenever $f : [-\pi, \pi] \to \mathbb{C}$ is Riemann integrable.

   (c) Prove that a Riemann integrable real-valued function $f : [-\pi, \pi] \to \mathbb{R}$ always has Fourier coefficients that satisfy $c_n = \overline{c_{-n}}$ for all $n \in \mathbb{Z}$. (Here the bar over c_{-n} represents complex conjugation.) What can we conclude about c_0?

3. Problem 1 on page 42 of Folland.
4. Problem 2 on page 42 of Folland.
5. Problem 5 on page 43 of Folland.
6. Problem 7 on page 43 of Folland.
7. Problem 5 on page 48 of Folland.
8. Problem 8 on page 48 of Folland.
9. Problem 11 on page 48 of Folland.
10. Problem 12 on page 48 of Folland.