CMSE 823, **Test 2** **November 3, 2016**

Name: ______________________________

Instructions:

- You only need a pen, or a pencil, your text book, notes, and an eraser.
- In particular, this means no calculator or any electronic devices – **only paper!**
- You must show appropriate legible work and justify your answer to receive full credit.
- Check and make sure there are five total pages including the cover page, when you begin the exam.
- You have 80 minutes to do the five problems below.

Good Luck!

Problem	Score	Out of
1		30
2		15
3		15
4		25
5		15
Total		100

1. **LU Factorizations:** Answer the following questions about $A \in \mathbb{R}^{4\times 4}$ below.

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{1001}{1000} & -2 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

Note: This problem has three subparts – check the next page!

(a) Compute the LDU decomposition of A without pivoting.

(b) Calculate the product of the eigenvalues of A, and state the facts that make this easy to do.

(c) Compute the growth factor for A based on your LDU decomposition from part (a). How much will this growth factor improve if you use partial pivoting? *There is no need to rewrite any calculations that don't change from part (a).*

2. **Positive Definite Matrices:** If $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite show that

$$|a_{i,j}| < \sqrt{a_{i,i}a_{j,j}}$$

holds for all $1 \leq i, j \leq n$. *Hint: Recall that B^*AB will also be positive definite for all full rank $B \in \mathbb{C}^{n \times m}$.*

3. **Approximate Diagonalization:** Suppose $A \in \mathbb{C}^{n \times n}$ is Hermitian. Describe how to construct a unitary matrix Q so that $Q^* A Q = T$ is real, symmetric, and tridiagonal *without knowing/using A's eigenvectors*. Demonstrate at least one step of the process using block matrix notation before describing how the k^{th} step would proceed.

4. **Condition Numbers, Operator Norms, and Singular Values.** This problem has three parts.

(a) Let $A \in \mathbb{C}^{n \times n}$ have columns $\mathbf{a}_j \in \mathbb{C}^n$ for each $j = 1, \ldots, n$. Prove that

$$\|A\|_p \geq \max_{1 \leq j \leq n} \|\mathbf{a}_j\|_p$$

holds for any (operator) p-norm, $1 \leq p \leq \infty$. *Hint: Consider the standard basis vectors* $\mathbf{e}_j$.

(b) Prove that the smallest singular value of A satisfies $\sigma_n(A) \leq \min_{1 \leq j \leq n} \|\mathbf{a}_j\|_2$.

(c) Given data $\mathbf{d} \in \mathbb{C}^4$, let $A \in \mathbb{C}^{4\times 4}$ be such that

$$A\mathbf{d} = \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}$$

where $a, b, c, e \in \mathbb{C}$ satisfy $d_j = a(j+9)^3 + b(j+9)^2 + c(j+9) + e$ for all $j = 1, 2, 3, 4$. Give the best **lower bound** you can for the condition number of A, $\kappa(A)$, assuming parts (a) and (b). **You do not have to simplify your answer!**

5. **Stability:** Let $f(c)$ output the smallest magnitude root of the quadratic polynomial

$$p(x) = \sqrt{\epsilon}x^2 + \pi x + c$$

so that $f : \mathbb{R} \to \mathbb{C}$ is given by

$$f(c) := \frac{-\pi \pm \sqrt{\pi^2 - 4c\sqrt{\epsilon}}}{2\sqrt{\epsilon}}.$$

Here the sign $\pm$ is chosen in order to get the smaller magnitude of the two roots from the quadratic formula, and $\epsilon = \epsilon_{\text{machine}} \in \mathbb{R}^+$.

(a) Consider c with $|c| \leq \epsilon$. What is the largest value that $|f(c + O(\epsilon))| = |f(O(\epsilon))|$ can have? Give your answer using big-O notation $O(\cdot)$.

(b) Let $\tilde{f}$ be defined by

$$\tilde{f}(c) = \mathrm{fl}\left(\frac{\mathrm{fl}\left(-\pi \pm \tilde{g}\left(\mathrm{fl}\left(\mathrm{fl}\left(\pi^2\right) - \mathrm{fl}\left(4 * c * \sqrt{\epsilon}\right)\right)\right)\right)}{\mathrm{fl}(2 * \sqrt{\epsilon})}\right) \quad \text{for } c \in F,$$

where $\tilde{g} : F \to F$ is a backward stable implementation of the square root function. Is $\tilde{f}$ backward stable? Is it stable? Explain.