STANDARD MONOMIAL THEORY FOR BOTT-SAMELSON VARIETIES

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Abstract. We construct a standard monomial basis for the space of sections $H^0(Z, \mathcal{L})$, where Z is a Bott-Samelson variety and $\mathcal{L}$ a positive line bundle over Z. As a special case, we recover and complete the classical Standard Monomial Theory for an arbitrary semisimple algebraic group.

THÉORIE DES MONÔMES STANDARD POUR LES VARIÉTÉS DE BOTT-SAMELSON

Résumé. Nous construisons une théorie des monômes standard pour l'espace des sections $H^0(Z, \mathcal{L})$, où Z est une variété de Bott-Samelson et où $\mathcal{L}$ est un fibré en droites positif sur Z. En particulier, nous retrouvons et complétons la théorie des monômes standard classique pour un groupe algébrique semisimple arbitraire.

Version française abrégée. Soient G un groupe algébrique semisimple défini sur un corps algébriquement clos de caractéristique arbitraire (ou sur $\mathbb{Z}$), W son groupe de Weyl engendré par les réflections simples $\{s_1, \ldots, s_r\}$, B un sous-groupe de Borel, $P_i \supset B$ le sous-groupe parabolique minimal associé à s_i, et $\widehat{P}_i \supset B$ le sous-groupe parabolique maximal associé à $\{s_1, \ldots, \widehat{s_i}, \ldots, s_r\}$. Notons par α_i les racines simples, et par ϖ_i les poids fondamentaux (voir [Bo, Ja]).

Choisissons une suite *arbitraire* de réflections simples $(s_{i_1}, s_{i_2}, \ldots, s_{i_l})$, que nous identifierons au mot $\mathbf{i} = (i_1, i_2, \cdots, i_l)$. La variété de Bott-Samelson associée [De, Ma2] est le quotient

$$Z_{\mathbf{i}} = P_{i_1} \times P_{i_2} \times \cdots \times P_{i_l} \ / \ B^l \ ,$$

où B^l opère à droite sur $P_{i_1} \times \cdots \times P_{i_l}$ par

$$(p_1, p_2, \ldots, p_l) \cdot (b_1, \ldots, b_l) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{l-1}^{-1} p_l b_l).$$

Soit $\mathbf{m} = (m_1, \ldots, m_l)$ avec $m_j \in \mathbb{Z}^+$. Définissons le plongement

$$\begin{array}{cccc} \iota : & Z_{\mathbf{i}} & \to & (G/\widehat{P}_{i_1})^{m_1} \times (G/\widehat{P}_{i_2})^{m_2} \times \cdots \times (G/\widehat{P}_{i_l})^{m_l} \overset{\text{def}}{=} X_{\mathbf{i}}^{\mathbf{m}} \\ & (p_1, p_2, \ldots, p_l) & \mapsto & (\underbrace{\overline{p_1}, \ldots, \overline{p_1}}_{m_1 \text{ fois}}, \underbrace{\overline{p_1 p_2}, \ldots, \overline{p_1 p_2}}_{m_2 \text{ fois}}, \ldots, \underbrace{\overline{p_1 \cdots p_l}, \ldots, \overline{p_1 \cdots p_l}}_{m_l \text{ fois}}) \end{array},$$

Key words and phrases. Young tableau, LS paths, crystal graphs, Schur modules.
Both authors partially supported by the National Science Foundation.

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où $\overline{p}$ désigne la classe de l'élément p.

Soit $\mathcal{O}(1)$ le fibré en droites ample de degré minimal sur $X_{\mathbf{i}}^{\mathbf{m}}$; sa restriction à $Z_{\mathbf{i}}$ est notée $\mathcal{L}_{\mathbf{m}} = \iota^*\mathcal{O}(1)$. On note encore $\mathcal{O}(1)$ le fibré ample minimal sur $G/\widehat{P}_i$.

Notre résultat principal construit une base de l'espace des sections globales $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$ formée de “monômes standard”, restrictions de certaines sections particulières dans l'espace

$$H^0(X_{\mathbf{i}}^{\mathbf{m}}, \mathcal{O}(1)) \;\cong\; V^*(\varpi_{i_1})^{\otimes m_1} \otimes \ldots \otimes V^*(\varpi_{i_l})^{\otimes m_l},$$

où $V^*(\varpi_i) \cong H^0(G/\widehat{P}_i, \mathcal{O}(1))$ désigne le dual de la i-ième représentation fondamentale de G (un module de Weyl dual, qui est irréductible sur un corps de caractéristique nulle, mais non en général).

Le premier auteur a construit une base [La] pour $V^*(\varpi_i)$, et en fait pour chaque $V^*(\lambda)$. (Plus précisément, la base donnée dans [La] est pour l'espace dual $V(\varpi_i)$, et ici nous considérons sa base duale). La base de $V^*(\varpi_i)$ est indexée par certaines suites de poids extrémaux et de nombres rationnels, les suites de Lakshmibai-Seshadri $\mathcal{LS}(\varpi_i)$ (voir [Li]). Par exemple, lorsque G est un groupe classique, les nombres rationnels sont superflus, et $\mathcal{LS}(\varpi_i)$ est formé des couples $\pi = (\tau, \tau')$ de poids $\tau = w(\varpi_i)$, $\tau' = w'(\varpi_i)$, pour lesquels il existe une suite $\tau = \tau_0, \tau_1, \ldots, \tau_q = \tau'$ telle que pour tout j, $\tau_{j+1} = s_k(\tau_j)$ pour un certain k et $\tau_{j+1} - \tau_j = 2\alpha_k$. (Littelmann identifie un tel couple avec un chemin linéaire par morceaux dans l'espace des poids, allant de 0 à $\frac{1}{2}\tau$ puis à $\frac{1}{2}\tau + \frac{1}{2}\tau'$.)

L'élément de la base de $V^*(\varpi_i)$ associé à π est noté $p_\pi^{\varpi_i}$. Ce base est compatible avec les variétés de Schubert dans $G/\widehat{P}_i$ (ce point est essentiel). Nous pouvons maintenant construire une base de $H^0(X_{\mathbf{i}}^{\mathbf{m}}, \mathcal{O}(1))$ indexée par les suites $\pi = (\pi_{11}, \ldots, \pi_{1m_1}, \pi_{21}, \ldots, \pi_{lm_l})$ avec $\pi_{km} \in \mathcal{LS}(\varpi_{i_k})$, et qui est formée des monômes $p_\pi = p_{\pi_{11}}^{\varpi_{i_1}} \otimes \cdots \otimes p_{\pi_{lm_l}}^{\varpi_{i_l}}$. (Nous pouvons encore identifier les suites π avec des chemins linéaires par morceaux, par concaténation comme dans [Li].)

Un monôme p_π est appelé *standard* si π possède un “relèvement $\mathbf{i}$-compatible”. Dans le cas des groupes classiques, cela signifie que pour $\pi = (\tau_{11}, \tau'_{11}, \ldots, \tau_{lm_l}, \tau'_{lm_l})$, il existe une suite de sous-mots de $\mathbf{i} = (i_1, \ldots, i_l)$ donnée par des sous-ensembles $\{1, \ldots, l\} \supset J_{11} \supset J'_{11} \supset \cdots \supset J_{lm_l} \supset J'_{lm_l}$ telle que pour tous k, m,

$$\tau_{km} = (\prod_{\substack{j \in J_{km} \\ j \leq k}} s_{i_j})(\varpi_{i_k}),$$

et de même pour τ'_{km} et J'_{km} (c.f. [LS]). Le poids d'une suite π (le caractère du tore maximal agissant sur l'élément associé p_π de la base) est l'opposé de l'extrémité du chemin associé: $-(\frac{1}{2}\tau_{11} + \frac{1}{2}\tau'_{11} + \ldots + \frac{1}{2}\tau'_{lm_l})$ (le signe moins est dû à la dualité). La définition pour un groupe G arbitraire est analogue, et en particulier elle ne fait jamais intervenir les nombres rationnels dans la suite de LS.

Théorème. *Les monômes standard p_π sur $X_{\mathbf{i}}^{\mathbf{m}}$ se restreignent en une base de $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$.*

Note. Parmi les fibrés en droites $\mathcal{L}_{\mathbf{m}}$ figurent les images inverses des fibrés en droites sur la variété de Schubert X_w où $w = s_{i_1} \cdots s_{i_l}$. Ainsi, le résultat précédent contient comme cas particulier la construction de bases pour les modules de Demazure $V_w^*(\lambda) \cong H^0(X_w, \mathcal{L}_\lambda)$, ce qui est le sujet de la théorie des monômes standard classique.

French Summary. Let G be a semi-simple algebraic group defined over an algebraically closed field of arbitrary characteristic (or over $\mathbb{Z}$), W its Weyl group generated by the simple reflections $\{s_1, \dots, s_r\}$, B a Borel subgroup, $P_i \supset B$ the minimal parabolic associated to s_i, and $\widehat{P}_i \supset B$ the maximal parabolic associated to $\{s_1, \dots, \widehat{s_i}, \dots, s_r\}$. Denote the simple roots by α_i and the fundamental weights by ϖ_i. (See [Bo, Ja].)

Choose an *arbitrary* sequence of simple reflections $(s_{i_1}, s_{i_2}, \dots, s_{i_l})$, which we identify with the word $\mathbf{i} = (i_1, i_2, \cdots, i_l)$. The corresponding Bott-Samelson variety [De, Ma2] is the quotient space

$$Z_{\mathbf{i}} = P_{i_1} \times P_{i_2} \times \cdots \times P_{i_l} \ / \ B^l \ ,$$

where B^l acts on the right of $P_{i_1} \times \cdots \times P_{i_l}$ by

$$(p_1, p_2, \dots, p_l) \cdot (b_1, \dots, b_l) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{l-1}^{-1} p_l b_l).$$

Let $\mathbf{m} = (m_1, \dots, m_l)$ with $m_j \in \mathbb{Z}^+$, and define the embedding

$$\begin{array}{rcl} \iota: \quad Z_{\mathbf{i}} & \rightarrow & (G/\widehat{P}_{i_1})^{m_1} \times (G/\widehat{P}_{i_2})^{m_2} \times \cdots \times (G/\widehat{P}_{i_l})^{m_l} \stackrel{\text{def}}{=} X_{\mathbf{i}}^{\mathbf{m}} \\ (p_1, p_2, \dots, p_l) & \mapsto & (\underbrace{\overline{p_1}, \dots, \overline{p_1}}_{m_1 \text{ times}}, \underbrace{\overline{p_1 p_2}, \dots, \overline{p_1 p_2}}_{m_2 \text{ times}}, \dots, \underbrace{\overline{p_1 \cdots p_l}, \dots, \overline{p_1 \cdots p_l}}_{m_l \text{ times}}) \end{array},$$

where $\overline{p}$ means the coset of the group element p in the appropriate G/P.

Let $\mathcal{O}(1)$ be the minimal-degree ample line bundle on $X_{\mathbf{i}}^{\mathbf{m}}$, and denote its restriction to $Z_{\mathbf{i}}$ by $\mathcal{L}_{\mathbf{m}} = \iota^* \mathcal{O}(1)$. We also denote by $\mathcal{O}(1)$ the minimal ample bundle on each $G/\widehat{P}_i$.

Our main result is to construct a basis of the global sections $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$ consisting of "standard monomials", the restrictions of certain special sections in the larger space

$$H^0(X_{\mathbf{i}}^{\mathbf{m}}, \mathcal{O}(1)) \ \cong \ V^*(\varpi_{i_1})^{\otimes m_1} \otimes \ldots \otimes V^*(\varpi_{i_l})^{\otimes m_l},$$

where $V^*(\varpi_i) \cong H^0(G/\widehat{P}_i, \mathcal{O}(1))$ denotes the dual of the i^{th} fundamental representation of G (a dual Weyl module, which is irreducible over a field of characteristic zero but not in general).

The first author has given a basis [La] for $V^*(\varpi_i)$, and in fact for any $V^*(\lambda)$. (More precisely, the basis given there is for the dual space $V(\lambda)$, so here we take the dual basis.) The basis of $V^*(\varpi_i)$ is indexed by certain sequences of extremal weights and rational numbers, the Lakshmibai-Seshadri sequences $\mathcal{LS}(\varpi_i)$ (see [Li]). For example, in the case of any classical group G the rational numbers are redundant, and $\mathcal{LS}(\varpi_i)$ consists of all pairs $\pi = (\tau, \tau')$ of weights $\tau = w(\varpi_i)$, $\tau' = w'(\varpi_i)$, for which there exists a sequence $\tau = \tau_0, \tau_1, \dots, \tau_q = \tau'$ such that for each j, $\tau_{j+1} = s_k(\tau_j)$ for some k and $\tau_{j+1} - \tau_j = 2\alpha_k$. (Littelmann identifies such a pair with a piecewise-linear path in the weight lattice going from 0 to $\frac{1}{2}\tau$ to $\frac{1}{2}\tau + \frac{1}{2}\tau'$.)

The basis element in $V^*(\varpi_i)$ corresponding to π is denoted $p_\pi^{\varpi_i}$. Crucially, this basis is compatible with the Schubert varieties in $G/\widehat{P}_i$. Now we can give a basis of $H^0(X_{\mathbf{i}}^{\mathbf{m}}, \mathcal{O}(1))$ indexed by sequences $\pi = (\pi_{11}, \dots, \pi_{1m_1}, \pi_{21}, \dots, \pi_{lm_l})$ with $\pi_{km} \in \mathcal{LS}(\varpi_{i_k})$, consisting of all monomials $p_\pi = p_{\pi_{11}}^{\varpi_{i_1}} \otimes \cdots \otimes p_{\pi_{lm_l}}^{\varpi_{i_l}}$. (We may once again identify sequences π with piecewise-linear paths by head-to-tail concatenation of the entries as in [Li].)

We call a monomial p_π *standard* if π possesses an "$\mathbf{i}$-compatible lifting". Again for classical groups, this means that for $\pi = (\tau_{11}, \tau'_{11}, \dots, \tau_{lm_l}, \tau'_{lm_l})$, there exists a chain of subwords of $\mathbf{i} = (i_1, \dots, i_l)$ given by subsets $\{1, \dots, l\} \supset J_{11} \supset J'_{11} \supset \cdots \supset J_{lm_l} \supset J'_{lm_l}$ such that for all k, m,

$$\tau_{km} = \Big(\prod_{\substack{j \in J_{km} \\ j \leq k}} s_{i_j} \Big)(\varpi_{i_k}),$$

and similarly for τ'_{km} and J'_{km} (c.f. [LS]). The weight of a sequence π (the character of the maximal torus acting on the corresponding basis element p_π) is negative the endpoint of its path: $-(\frac{1}{2}\tau_{11} + \frac{1}{2}\tau'_{11} + \ldots + \frac{1}{2}\tau'_{lm_l})$. (The minus sign is due to dualization.) The definition for a general G is similar, and in particular never involves the rational-number data in the LS sequence.

Theorem. *The standard monomials p_π on $X_{\mathbf{i}}^{\mathbf{m}}$ restrict to a basis of $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$.*

Note. The line bundles $\mathcal{L}_{\mathbf{m}}$ include those pulled back from the Schubert variety X_w, where $w = s_{i_1} \cdots s_{i_l}$, so the above theorem includes as a special case the construction of bases for the (dual) Demazure modules $V_w^*(\lambda) \cong H^0(X_w, \mathcal{L}_\lambda)$, which is the subject of classical Standard Monomial Theory [LS].

English Version. For ease of exposition, we suppose G to be of type A_{n-1}. Let $G = SL(n, \mathbb{F})$ (where $\mathbb{F}$ is an algebraically closed field of arbitrary characteristic or $\mathbb{F} = \mathbb{Z}$); consider the subgroups B = upper triangular matrices, P_i = matrices which are upper triangular except for the position $(i+1, i)$, W = permutation matrices; let s_i = the transposition $(i, i+1)$; and note that $G/\widehat{P}_i = \mathrm{Gr}(i, \mathbb{F}^n) = \mathrm{Gr}(i)$ the Grassmannian of i-planes in n-space.

For any word $\mathbf{i} = (i_1, \dots, i_l)$, $1 \leq i_j \leq n-1$, reduced or non-reduced, and any sequence $\mathbf{m} = (m_1, \dots, m_l)$, $m_j \in \mathbb{Z}^+$, the Bott-Samelson variety

$$Z_{\mathbf{i}} = P_{i_1} \overset{B}{\times} \cdots \overset{B}{\times} P_{i_l}/B$$

embeds into the product of Grassmannians

$$\mathrm{Gr}(\mathbf{i})^{\mathbf{m}} = X_{\mathbf{i}}^{\mathbf{m}} = \mathrm{Gr}(i_1)^{m_1} \times \cdots \times \mathrm{Gr}(i_l)^{m_l}$$

by

$$\begin{array}{cccc} \iota: & Z_{\mathbf{i}} & \to & \mathrm{Gr}(\mathbf{i})^{\mathbf{m}} \\ & (p_1,\ldots,p_l) & \mapsto & (\underbrace{p_1\mathbb{F}^{i_1},\ldots,p_1\mathbb{F}^{i_1}}_{m_1 \text{ times}},\ldots,\underbrace{p_1\cdots p_l\mathbb{F}^{i_l},\ldots,p_1\cdots p_l\mathbb{F}^{i_l}}_{m_l \text{ times}}) \end{array}$$

where $0 \subset \mathbb{F}^1 \subset \cdots \subset \mathbb{F}^n$ is the standard flag.

Now, each Grassmannian has a minimal-degree ample line bundle $\mathcal{O}(1)$, and the tensor product of these for each i_j is the minimal ample line bundle on $\mathrm{Gr}(\mathbf{i})^{\mathbf{m}}$ which we again denote $\mathcal{O}(1) \stackrel{\text{def}}{=} \mathcal{O}(1,\ldots,1)$. Denote its restriction to $Z_{\mathbf{i}}$ by $\mathcal{L}_{\mathbf{m}} = \iota^*\mathcal{O}(1)$.

Problem. *Find an explicit basis for* $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$.

Note. The line bundles $\mathcal{L}_{\mathbf{m}}$ include those pulled back from the Schubert variety X_w, where $w = s_{i_1}\cdots s_{i_l}$, so the above problem includes as a special case the construction of bases for the (dual) Demazure modules $V_w^*(\lambda) \cong H^0(X_w, \mathcal{L}_\lambda)$, which is the subject of classical Standard Monomial Theory [LS].

We first consider the fundamental Weyl module $H^0(\mathrm{Gr}(i), \mathcal{O}(1)) \cong V^*(\varpi_i)$. This has a well-known basis consisting of the Plücker coordinates,

$$V^*(\varpi_i) = \mathrm{Span}_{\mathbb{F}}\left\{\Delta_I \mid I \subset [n] \text{ and } |I| = i\right\},$$

where $[n] = \{1, 2, \ldots, n\}$. Here Δ_I denotes the $i \times i$ minor on the rows I of the $n \times i$ matrix of homogeneous coordinates on the Grassmannian $\mathrm{Gr}(i)$. Thus, we get a basis of

$$\begin{aligned} H^0(\mathrm{Gr}(\mathbf{i})^{\mathbf{m}}, \mathcal{O}(1)) &\cong V^*(\varpi_{i_1})^{\otimes m_1} \otimes \cdots \otimes V^*(\varpi_{i_l})^{\otimes m_l} \\ &= \mathrm{Span}\left\{\Delta_\pi \;\middle|\; \begin{array}{l} \pi = (I_{11},\ldots,I_{1m_1}, I_{21},\ldots,I_{lm_l}) \\ \forall\, k, m,\ I_{km} \subset [n] \text{ and } |I_{km}| = i_k \end{array}\right\}, \end{aligned}$$

where $\Delta_\pi = p_\pi = \Delta_{I_{11}} \otimes \cdots \otimes \Delta_{I_{lm_l}}$. We call Δ_π a *monomial* and π a *tableau*. We may again picture tableaux as paths.

Example. For $G = SL(3)$, $\mathbf{i} = (1,2,1)$, $m = (1,1,1)$, the tableau $\pi = (2,13,3) = 2.13.3$ is identified with the piecewise-linear path in $\mathbb{Z}^3$ with vertices $0, e_2, e_2+e_1+e_3, e_2+e_1+e_3+e_3$, where e_j denotes a coordinate vector. The corresponding basis element is

$$\Delta_\pi = \Delta_2(x)\,\Delta_{13}(y)\,\Delta_3(z) = x_2\,(y_{11}y_{32} - y_{31}y_{12})\,z_3$$

in the coordinates

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathrm{Gr}(1) \times \mathrm{Gr}(2) \times \mathrm{Gr}(1).$$

Definition. Let $[k] = \{1, 2, \ldots, k\}$, and allow W to act elementwise on subsets of $[n]$. We say a tableau π is *standard* if it possesses a "compatible lifting" with respect to **i**. That is, for $\pi = (I_{11}, \ldots, I_{lm_l})$, there exists a decreasing sequence of subwords of $\mathbf{i} = (i_1, \ldots, i_l)$, indexed by subsets $\{1, \ldots, l\} \supset J_{11} \supset \cdots \supset J_{lm_l}$, such that for all k, m,

$$(\prod_{\substack{j \in J_{km} \\ j \leq k}} s_{i_j})[i_k] = I_{km}.$$

A monomial is standard if its defining tableau is standard.

Note. There is an efficient recursive algorithm for generating the standard tableaux, given by a refined Demazure character formula based on Littelmann's path operators [Li] (first defined for $SL(n)$ by Lascoux and Schützenberger [LSch]).

Theorem. *The standard monomials* Δ_π *on* $\mathrm{Gr}(\mathbf{i})^{\mathbf{m}}$ *restrict to a basis of* $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$.

Example. In the above case, $\pi = 2.13.3$ is standard since the sequence $J_{11} = \{1,2,3\}, J_{21} = \{2,3\}, J_{31} = \{2,3\}$ gives $s_{i_1}[i_1] = s_1[1] = 2$, $s_{i_2}[i_2] = s_2[2] = 13$, and $s_{i_2}s_{i_3}[i_3] = s_2 s_1[1] = 3$. There are thirteen standard tableaux: 1.12.1, 1.12.2, 1.13.2, 1.13.3, 2.13.2, 2.13.3, 2.23.3, 1.13.1, 2.12.1, 2.12.2, 2.13.1, 2.23.1, 2.23.2.

Sketch of Proof. We prove the Theorem by induction on the dimension of $Z_{\mathbf{i}}$ (= the length of **i**) and on the degree of $\mathcal{L}_{\mathbf{m}}$. We show that both the space of sections and the span of the standard monomials can be built up using the same recurrences, and hence they must be identical spaces.

Let $\mathcal{T}(\mathbf{i}, \mathbf{m})$ denote the set of standard tableaux for the space $H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$. Suppose $\mathbf{m} = (m_1, \ldots, m_l)$ and let k be the smallest value with $m_k > 0$. Without loss of generality, we may assume that the initial subword $(i_1, \ldots, i_k)$ is reduced. That is, let $\widetilde{\mathbf{i}}$ be a subword of **i** with some of the first k letters removed so as to make $(i_1, \ldots, i_k)$ reduced. Then we can show that $H^0(Z_{\widetilde{\mathbf{i}}}, \mathcal{L}_{\mathbf{m}}) \cong H^0(Z_{\mathbf{i}}, \mathcal{L}_{\mathbf{m}})$ and that $\mathcal{T}(\widetilde{\mathbf{i}}, \mathbf{m}) = \mathcal{T}(\mathbf{i}, \mathbf{m})$.

Next, let $Z = Z_{\mathbf{i}}$ and for $1 \leq j \leq l$, let $Z_{\hat{\jmath}}$ denote the Bott-Samelson variety of the word **i** with the j^{th} entry removed, $\mathbf{i} - i_j = (i_1, \ldots, \widehat{i_j}, \ldots, i_l)$. Then $Z_{\hat{\jmath}}$ embeds naturally as a divisor in Z. Further, let $\mathcal{L}_{\epsilon(k)}$ denote the line bundle $\mathcal{L}_{\mathbf{m}}$ with $\mathbf{m} = \epsilon(k) = (0, \ldots, 1, \ldots, 0)$ (the 1 in the k^{th} place).

Now define $J(k) = s_{i_1} \cdots s_{i_k}[i_k] \subset [n]$. Then the work of Berenstein, Fomin, and Zelevinsky [BFZ, BZ] implies that

$$(\text{zero set of } \Delta_{J(k)}) \cap Z = \bigcup_{j \in \mathrm{Ess}(k)} Z_{\hat{\jmath}} \qquad \text{(scheme theoretically)}$$

where

$$\mathrm{Ess}(k) = \{ j \leq k \mid s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_k}[i_k] \neq J(k) \}.$$

Since $\Delta_{J(k)}$ is a section of $\mathcal{L}_{\epsilon(k)}$, this (together with a Kempf-type vanishing theorem for Z due to Mathieu [Mat1],[Mat2]) leads to the exact sequence

$$0 \to H^0(Z, \mathcal{L}_{\mathbf{m}} \otimes \mathcal{L}_{\epsilon(k)}^{-1}) \xrightarrow{\Delta_{J(k)}} H^0(Z, \mathcal{L}_{\mathbf{m}}) \xrightarrow{\text{rest}} H^0(\bigcup_{j \in \mathrm{Ess}(k)} Z_{\hat{\jmath}}, \mathcal{L}_{\mathbf{m}}) \to 0,$$

where the left-hand map is multiplication by the section $\Delta_{J(k)}$, and the right-hand map is restriction.

We can show the corresponding recurrence for the tableaux:

$$\mathcal{T}(\mathbf{i},\mathbf{m}) = J(k)\,.\,\mathcal{T}(\mathbf{i},\mathbf{m}-\epsilon(k)) \quad \sqcup \bigcup_{j\in \mathrm{Ess}(k)} \mathcal{T}(\mathbf{i}-i_j,\mathbf{m}),$$

where $J(k)\,.\,\mathcal{T}$ denotes the concatenation of $J(k)$ at the beginning of all the tableaux in $\mathcal{T}$, and $\mathcal{T}(\mathbf{i}-i_k,\mathbf{m}) \stackrel{\text{def}}{=} \mathcal{T}(\mathbf{i}-i_k,\mathbf{m}')$ for a suitable $\mathbf{m}'=(m'_1,\ldots,m'_{l-1})$.

Finally, the sections over the union $\cup_j Z_{\hat{j}}$ are analyzed via the Mayer-Vietoris sequence, and the tableaux again satisfy the same recurrence (provided we define a monomial to be standard on a union if it is standard on one of the components).

From this the Theorem follows immediately by induction. Namely, the standard monomials in $\Delta_{J(k)}\cdot H^0(Z,\mathcal{L}_{\mathbf{m}-\epsilon(k)})$ form a basis of this space by induction on the degree of $\mathcal{L}_{\mathbf{m}}$; and the standard monomials in $H^0(\cup_j Z_{\hat{j}},\mathcal{L}_{\mathbf{m}})$ form a basis of this space by induction on dimension; and the two give complementary subspaces of $H^0(Z,\mathcal{L}_{\mathbf{m}})$ by the exact sequence. Hence the standard monomials in $H^0(Z,\mathcal{L}_{\mathbf{m}})$ are a basis.

We thus show the Theorem not only for Bott-Samelson varieties, but also for the unions $\cup_{j\in\mathrm{Ess}(k)} Z_{\hat{j}}$. Note, however, that it does not hold for arbitrary unions even for the case of $SL(3)$.

The Theorem and proof are carried out similarly for a general reductive G in the spirit of [LS], and one may also extend the analysis to the case of symmetrizable Kac-Moody groups and quantum groups.

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