DEGENERACY SCHEMES, QUIVER SCHEMES AND SCHUBERT VARIETIES

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ABSTRACT. A result of Zelevinsky states that an orbit closure in the space of representations of the equioriented quiver of type A_h is in bijection with the opposite cell in a Schubert variety of a partial flag variety $SL(n)/P$. We prove that Zelevinsky's bijection is a scheme-theoretic isomorphism, which shows that the degeneracy schemes of Fulton and Buch are reduced and Cohen-Macaulay in arbitrary characteristic.

Among all algebraic varieties, the best understood are the flag varieties and their Schubert subvarieties. They first appear as interesting examples, but acquire a general importance in the theory of characteristic classes of vector bundles.

Fulton [8] and Buch-Fulton [6] have recently given a theory of "universal degeneracy loci", characteristic classes associated to maps among vector bundles, in which the role of Schubert varieties is taken by certain degeneracy schemes. The underlying varieties of these schemes arise in the theory of quivers: they are the orbit-closures in the space of representations of the equioriented quiver A_h. Many other classical varieties also appear as such quiver varieties, such as determinantal varieties and the variety of complexes (cf. §1.4). The same quiver varieties also arise in Deligne-Langlands theory for the p-adic general linear group [19]: the intersection homology of these varieties gives the p-adic analog of Kazhdan-Lusztig polynomials (which, by Zelevinsky's result below, become identical to ordinary Kazhdan-Lusztig polynomials).

It turns out that a separate theory is not necessary to understand these spaces (for this particular quiver). By a remarkable but little-known result of Zelevinsky [20], all the above quiver varieties can be identified set-theoretically with open subsets of Schubert varieties. In this paper, we prove a scheme-theoretic strengthening of Zelevinksy's identification: the "naive" determinantal conditions defining each quiver variety generate the same ideal as the Plucker equations defining the corresponding Schubert variety. Since the latter ideal is well understood via Standard Monomial Theory, we conclude that the corresponding quiver schemes are reduced and their singularities are identical to those of Schubert varieties. In particular, the quiver varieties in arbitrary characteristic are normal, Cohen-Macaulay, etc. These properties give a more concrete interpretation to the intersection theory in Fulton and Buch's work.

Our results extend early work by Hochster-Eagon [10], Kempf [12], and Deconcini-Strickland [7]. Musili-Seshadri [16], proved the above scheme-theoretic identification for the variety of complexes. Some of the consequences of our identification were known for more general quiver varieties by work of Abeasis, Del

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Fra, and Kraft [3],[1]: that the quiver varieties are Cohen-Macaulay with rational singularities over a field of characteristic zero, and that the determinantal conditions generate the reduced ideals of the quiver varieties of codimension one. Our methods are similar to those of Gonciulea and Lakshmibai [9].

1 Zelevinsky's bijection

In this section we establish the set-theoretic identification between quiver varieties and Schubert varieties. In §1.4, we give several examples, including Fulton's degeneracy schemes.

1.1 Quiver varieties

For the basic results below on quivers, we follow Abeasis-del Fra [2] and Zelevinsky [19]. Fix an h-tuple of non-negative integers $\mathbf{n} = (n_1, \ldots, n_h)$ and a list of vector spaces $V_1, \ldots, V_h$ over an arbitrary field $\mathbf{k}$ with respective dimensions $n_1, \ldots, n_h$. Define Z, the *variety of quiver representations* (of dimension $\mathbf{n}$, of the equioriented quiver of type A_h) to be the affine space of all $(h-1)$-tuples of linear maps $(f_1, \ldots, f_{h-1})$:

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{h-2}} V_{h-1} \xrightarrow{f_{h-1}} V_h \ .$$

If we endow each V_i with a basis, we get $V_i \cong \mathbf{k}^{n_i}$ and

$$Z \cong M(n_2 \times n_1) \times \cdots \times M(n_h \times n_{h-1}),$$

where $M(l \times m)$ denotes the affine space of matrices over $\mathbf{k}$ with l rows and m columns. The group

$$G_{\mathbf{n}} = GL(n_1) \times \cdots \times GL(n_h)$$

acts on Z by

$$(g_1, g_2, \cdots, g_h) \cdot (f_1, f_2, \cdots, f_{h-1}) = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \cdots, g_h f_{h-1} g_{h-1}^{-1}),$$

corresponding to change of basis in the V_i.

Now, let $\mathbf{r} = (r_{ij})_{1 \leq i \leq j \leq h}$ be an array of non-negative integers with $r_{ii} = n_i$, and define $r_{ij} = 0$ for any indices other than $1 \leq i \leq j \leq h$. Define the set

$$Z^\circ(\mathbf{r}) = \{(f_1, \cdots, f_{h-1}) \in Z \ \mid \ \forall\, i < j,\ \operatorname{rank}(f_{j-1} \cdots f_i : V_i \to V_j) = r_{ij}\}.$$

(This set might be empty for a bad choice of $\mathbf{r}$.)

Proposition. *The $G_{\mathbf{n}}$-orbits of Z are exactly the sets $Z^\circ(\mathbf{r})$ for $\mathbf{r} = (r_{ij})$ with*

$$r_{ij} - r_{i,j+1} - r_{i-1,j} + r_{i-1,j+1} \geq 0, \quad \forall\ 1 \leq i \leq j \leq h.$$

Proof. This is a standard result of algebraic quiver theory [5], [4], first stated in this form by Abeasis-del Fra and Zelevinsky. Since this theory is not well known among geometers, we recall it here.

Consider the abelian category $\mathcal{R}$ of quiver representations defined as follows. An object of $\mathcal{R}$ is a sequence of linear maps $(V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{h-1}} V_h)$, where the V_i are *any* vector spaces of *arbitrary* dimension. A morphism of $\mathcal{R}$ from the object $(V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{h-1}} V_h)$ to the object $(V_1' \xrightarrow{f_1'} \cdots \xrightarrow{f_{h-1}'} V_h')$ is defined to be an h-tuple of linear maps $(\phi_i : V_i \to V_i')$ such that each of the following squares commutes:

$$\begin{array}{ccc} V_i & \xrightarrow{f_i} & V_{i+1} \\ {\scriptstyle\phi_i} \downarrow & & \downarrow {\scriptstyle\phi_{i+1}} \\ V_i' & \xrightarrow{f_i'} & V_{i+1}' \end{array}$$

Direct sum of objects is defined componentwise, and it is known by the Krull-Schmidt Theorem [4] that any object $R \in \mathcal{R}$ can be written uniquely as a direct sum of indecomposable objects. By elementary linear algebra, these indecomposables are seen to be

$$R_{ij} = (0 \to \cdots \to 0 \to \underset{V_i}{\mathbf{k}} \xrightarrow{\sim} \cdots \xrightarrow{\sim} \underset{V_j}{\mathbf{k}} \to 0 \to \cdots \to 0)$$

for $1 \le i \le j \le h$ (corresponding to the positive roots of the root system A_h). That is, there are unique multiplicities $m_{ij} \in \mathbf{Z}^+$ with

$$R \cong \bigoplus_{1 \le i \le j \le h} m_{ij} R_{ij}.$$

Our variety Z consists of representations with fixed (V_i) and all possible (f_i). Two points of Z are in the same $G_{\mathbf{n}}$-orbit exactly if they are isomorphic as objects in $\mathcal{R}$. So the orbits correspond to arrays $(m_{ij})_{1 \le i \le j \le h}$ with $m_{ij} \in \mathbf{Z}^+$ and $n_i = \sum_{k \le i \le l} m_{kl}$.

We can compute the rank numbers $\mathbf{r} = (r_{ij})$ from the multiplicities $\mathbf{m} = (m_{ij})$:

$$r_{ij} = \sum_{k \le i \le j \le l} m_{kl},$$

and conversely

$$m_{ij} = r_{ij} - r_{i,j+1} - r_{i-1,j} + r_{i-1,j+1}$$

Hence the arrays (r_{ij}) with the stated conditions classify the $G_{\mathbf{n}}$-orbits on Z. •

We define the *quiver variety* as the algebraic set

$$Z(\mathbf{r}) = \{(f_1, \cdots, f_{h-1}) \in Z \mid \forall i,j,\ \operatorname{rank}(f_{j-1} \cdots f_i : V_i \to V_j) \le r_{ij}\}.$$

It will follow from Zelevinsky's theorem (§1.3) that $Z(\mathbf{r})$ is an irreducible variety and is the Zariski closure of $Z^\circ(\mathbf{r})$ (provided the base field $\mathbf{k}$ is infinite).

1.2 Schubert varieties

Given $\mathbf{n} = (n_1, \cdots, n_h)$, for $1 \le i \le h$ let

$$a_i = n_1 + n_2 + \cdots + n_i, \qquad \text{and} \qquad n = n_1 + \cdots + n_h \ .$$

For positive integers $i \le j$, we shall frequently use the notations

$$[i,j] = \{i, i+1, \ldots, j\}, \qquad [i] = [1,i], \qquad [0] = \{\}\ .$$

Let $\mathbf{k}^n \cong V_1 \oplus \cdots \oplus V_h$ have basis $e_1, \ldots, e_n$ compatible with the V_i. Consider its general linear group $GL(n)$, the subgroup B of upper-triangular matrices, and the parabolic subgroup P of block upper-triangular matrices

$$P = \{(a_{ij}) \in GL(n) \mid a_{ij} = 0 \text{ whenever } j \le a_k < i \text{ for some } k\}\ .$$

A *partial flag of type* $(a_1 < a_2 < \cdots < a_h = n)$ (or simply a *flag*) is a sequence of subspaces $U_\bullet = (U_1 \subset U_2 \subset \cdots \subset U_h = \mathbf{k}^n)$ with $\dim U_i = a_i$. Let $E_i = V_1 \oplus \cdots \oplus V_i = \langle e_1, \ldots, e_{a_i}\rangle$, and $E'_i = V_{i+1} \oplus \cdots \oplus V_h = \langle e_{a_i+1}, \ldots, e_n\rangle$, so that $E_i \oplus E'_i = \mathbf{k}^n$. The *flag variety* Fl is the set of all flags $U_\bullet$ as above.

Fl has a transitive $GL(n)$-action induced from $\mathbf{k}^n$, and $P = \mathrm{Stab}_{GL(n)}(E_\bullet)$, so we may identify $\mathrm{Fl} \cong GL(n)/P$, $gE_\bullet \leftrightarrow gP$. The *Schubert varieties* are the closures of B-orbits on Fl. Such orbits are usually indexed by certain permutations of $[n]$, but we prefer to use *flags of subsets* of $[n]$, of the form

$$\tau = (\tau_1 \subset \tau_2 \subset \cdots \subset \tau_h = [n]), \qquad \#\tau_i = a_i\ .$$

(A permutation $w : [n] \to [n]$ corresponds to the subset-flag with $\tau_i = w[a_i] = \{w(1), w(2), \ldots, w(a_i)\}$. This gives a one-to-one correspondence between cosets of the symmetric group $W = S_n$ modulo the Young subgroup $W_{\mathbf{n}} = S_{n_1} \times \cdots \times S_{n_h}$, and subset-flags.)

Given such τ, let $E_i(\tau) = \langle e_j \mid j \in \tau_i\rangle$ be a coordinate subspace of $\mathbf{k}^n$, and $E_\bullet(\tau) = (E_1(\tau) \subset E_2(\tau) \subset \cdots) \in \mathrm{Fl}$. Then we may define the *Schubert cell*

$$\begin{aligned} X^\circ(\tau) &= B \cdot E(\tau) \\ &= \left\{ (U_1 \subset U_2 \subset \cdots) \in \mathrm{Fl} \;\middle|\; \begin{array}{c} \dim U_i \cap \mathbf{k}^j = \#\,\tau_i \cap [j] \\ 1 \le i \le h,\ 1 \le j \le n \end{array} \right\} \end{aligned}$$

and the *Schubert variety*

$$\begin{aligned} X(\tau) &= \overline{X^\circ(\tau)} \\ &= \left\{ (U_1 \subset U_2 \subset \cdots) \in \mathrm{Fl} \;\middle|\; \begin{array}{c} \dim U_i \cap \mathbf{k}^j \ge \#\,\tau_i \cap [j] \\ 1 \le i \le h,\ 1 \le j \le n \end{array} \right\} \end{aligned}$$

where $\mathbf{k}^j = \langle e_1, \ldots, e_j\rangle \subset \mathbf{k}^n$.

We define the *opposite cell* $\mathbf{O} \subset \mathrm{Fl}$ to be the set of flags in general position with respect to the spaces $E'_1 \supset \cdots \supset E'_{h-1}$:

$$\mathbf{O} = \{(U_1 \subset U_2 \subset \cdots) \in \mathrm{Fl} \ \mid\ U_i \cap E'_i = 0\}.$$

In fact, $\mathbf{O} = B_- \cdot E_\bullet$, the orbit of the standard flag in Fl under the group B_- of lower triangular matrices. We also define $Y(\tau) = X(\tau) \cap \mathbf{O}$, an open subset of $X(\tau)$ and $Y^\circ(\tau) = X^\circ(\tau) \cap \mathbf{O}$. By abuse of language, we call $Y(\tau)$ the *opposite cell* of $X(\tau)$, even though it is not a cell.

1.3 The bijection ζ

We define a special subset-flag $\tau^{\max} = (\tau_1^{\max} \subset \cdots \subset \tau_h^{\max} = [n])$ corresponding to $\mathbf{n} = (n_1, \ldots, n_h)$. We want each $\tau_i^{\max}$ to contain numbers as large as possible given the constraints $[a_{j-1}] \subset \tau_j^{\max}$ for all j (here $a_0 = 0$). Namely, we define $\tau_i^{\max}$ recursively by

$$\tau_h^{\max} = [n]; \quad \tau_i^{\max} = [a_{i-1}] \cup \{\text{largest } n_i \text{ elements of } \tau_{i+1}^{\max}\}.$$

Furthermore, given $\mathbf{r} = (r_{ij})_{1 \le i \le j \le h}$ indexing a quiver variety, define a subset-flag $\tau^{\mathbf{r}}$ to contain numbers as large as possible given the constraints

$$\#\, \tau_i^{\mathbf{r}} \cap [a_j] = \begin{cases} a_i - r_{i,j+1} & \text{for } i \le j \\ a_j & \text{for } i > j \end{cases}$$

Namely,

$$\tau_i^{\mathbf{r}} = \{ \underbrace{1 \ldots a_{i-1}}_{a_{i-1}} \; \underbrace{\ldots\ldots a_i}_{r_{ii} - r_{i,i+1}} \; \underbrace{\ldots\ldots a_{i+1}}_{r_{i,i+1} - r_{i,i+2}} \; \underbrace{\ldots\ldots a_{i+2}}_{r_{i,i+2} - r_{i,i+3}} \; \cdots \; \underbrace{\ldots\ldots n}_{r_{i,h}} \}$$

where we use the visual notation

$$\underbrace{\cdots\cdots a}_{b} = [a - b + 1, a].$$

Recall that $a_j = a_{j-1} + n_j$ and $0 \le r_{ij} - r_{i,j+1} \le n_j$, so that each $\tau_i^{\mathbf{r}}$ is an increasing list of integers. Also $r_{ij} - r_{i,j+1} \le r_{i+1,j} - r_{i+1,j+1}$, so that $\tau_i^{\mathbf{r}} \subset \tau_{i+1}^{\mathbf{r}}$. Thus $\tau^{\mathbf{r}}$ is indeed a subset-flag. See §1.4 for examples.

Now define the Zelevinsky map

$$\begin{array}{cccc} \zeta : & Z & \to & \mathrm{Fl} \\ & (f_1, \ldots, f_{h-1}) & \mapsto & (U_1 \subset U_2 \subset \cdots) \end{array}$$

where

$$U_i = \{(v_1, \ldots, v_h) \in V_1 \oplus \cdots \oplus V_h = \mathbf{k}^n \mid \forall\, j \ge i,\; v_{j+1} = f_j(v_j)\}.$$

In terms of coordinates, if we identify the linear maps $(f_1, \ldots, f_{h-1})$ with the matrices $(A_1, \ldots, A_{h-1})$, and identify $\mathrm{Fl} \cong GL(n)/P$, we have

$$\zeta(A_1, \ldots, A_{h-1}) = \begin{pmatrix} I_1 & 0 & 0 & 0 & \cdots \\ A_1 & I_2 & 0 & 0 & \cdots \\ A_2 A_1 & A_2 & I_3 & 0 & \cdots \\ A_3 A_2 A_1 & A_3 A_2 & A_3 & I_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \mod P$$

where I_i is an identity matrix of size n_i.

Theorem. *(Zelevinsky [20])*
(i) ζ is a bijection of Z onto its image $Y(\tau^{max})$: $\quad \zeta : Z \xrightarrow{\sim} Y(\tau^{max})$.
Also,

$$(*) \qquad Y(\tau^{max}) = \{(U_1 \subset U_2 \subset \cdots) \;\mid\; \forall\ i, \quad E_{i-1} \subset U_i, \quad U_i \cap E'_i = 0\}.$$

(ii) ζ restricts to a bijection from $Z(\mathbf{r})$ onto $Y(\tau^{\mathbf{r}})$: $\quad \zeta : Z(\mathbf{r}) \xrightarrow{\sim} Y(\tau^{\mathbf{r}})$.
Also,

$$(**) \quad Y(\tau^{\mathbf{r}}) = \left\{ (U_1 \subset U_2 \subset \cdots) \;\middle|\; \begin{array}{c} \forall\ i \le j, \quad \dim U_i \cap E_j \ge a_i - r_{i,j+1}, \\ E_{i-1} \subset U_i, \quad U_i \cap E'_i = 0 \end{array} \right\} .$$

Proof. Obviously ζ is injective. To prove (i), we first show equation $(*)$. The inclusion $\subset$ is clear. For the inclusion $\supset$, consider a flag $U_\bullet$ with $E_{i-1} \subset U_i$ for all i. Since acting by B does not change $\dim U_i \cap \mathbf{k}^j$, we may suppose $U_\bullet = E_\bullet(\mu)$ for some $\mu = (\mu_1 \subset \cdots \subset \mu_h = [n])$ with $[a_{i-1}] \subset \mu_i$ for all i. By definition $\#\tau_i^{\max} \cap [j]$ is as small as possible given $[a_{i-1}] \subset \tau_i^{\max}$, so

$$\dim U_i \cap \mathbf{k}^j = \#\mu_i \cap [j] \ge \#\tau_i^{\max} \cap [j],$$

which shows $(*)$.

Now we show that $\zeta(Z)$ is equal to the right hand side of $(*)$. The inclusion $\zeta(Z) \subset \mathrm{RHS}(*)$ is clear, so we show $\zeta(Z) \supset \mathrm{RHS}(*)$. Each U_i is tansverse to E'_i, so U_i is the graph of a linear map

$$(f_{i,i+1}, \ldots, f_{i,h}) : E_i \to E'_i = V_{i+1} \oplus \cdots \oplus V_h.$$

Since $E_{i-1} \subset U_i$, we have $f_{ij}(E_{i-1}) = 0$, and we may consider $f_{ij} : V_i \cong E_i/E_{i-1} \to V_j$. Any element of U_j can be written $(v_1, \ldots, v_j, f_{j,j+1}(v_j), \ldots)$. Let $i < j$. Any $(v_1, \ldots, v_h) \in U_i$ is also an element of U_j, so $v_{j+1} = f_{j,j+1}(v_j)$. Taking $f_i = f_{i,i+1}$, we find $U_\bullet = \zeta(f_1, \ldots, f_{h-1})$.

The proof of (ii) is similar. Equation $(**)$ follows just as before. Now consider a flag $U_\bullet = \zeta(f_1 \ldots f_{h-1}) \in \zeta(Z) = Y(\tau^{\max})$. Then

$$\begin{aligned} \dim\, U_i \cap E_j &= \dim E_{i-1} + \dim \mathrm{Ker}(f_j f_{j-1} \cdots f_i) \\ &= \dim E_{i-1} + \dim V_i - \mathrm{rank}(f_j f_{j-1} \cdots f_i) \\ &= a_i - \mathrm{rank}(f_j f_{j-1} \cdots f_i). \end{aligned}$$

Hence $U_\bullet \in \zeta(Z(\mathbf{r})) \;\Leftrightarrow\; U_\bullet \in \mathrm{RHS}(**) = Y(\tau^{\mathbf{r}})$. $\bullet$

Corollary. ([2], [19]). For each $\mathbf{r}$, $Z^\circ(\mathbf{r})$ is an open dense $G_{\mathbf{n}}$-orbit in $Z(r)$.

Proof. Arguing as in the proof of (ii) above, we find that $\zeta(Z^\circ(\mathbf{r})) \supset Y^\circ(\tau^{\mathbf{r}})$. But it is known that $Y^\circ(\tau^{\mathbf{r}})$ is Zariski open and dense in $Y(\tau^{\mathbf{r}})$. $\bullet$

Remarks. (i) The map ζ is an algebraic isomorphism onto its image, since it is clear from the coordinate definition that ζ is injective on points and on tangent vectors.
(ii) For each $\mathbf{r}$, $X^\circ(\tau^{\mathbf{r}})$ is an orbit of P. If we embed $G_{\mathbf{n}}$ into P as block-diagonal matrices, then ζ is a $G_{\mathbf{n}}$-equivariant map. Now, clearly $\zeta(Z^\circ(\mathbf{r})) \subset Y^\circ(\tau^{\mathbf{r}})$. Also the $Z^\circ(\mathbf{r})$ are a complete list of $G_{\mathbf{n}}$-orbits on Z and the $Y^\circ(\tau^{\mathbf{r}})$ are disjoint subsets of $Y(\tau^{\max}) \cong \zeta(Z)$. We conclude that $\zeta(Z^\circ(\mathbf{r})) = Y^\circ(\tau^{\mathbf{r}})$.

1.4 Examples

Example. A small generic case. Let $h = 4$, $\mathbf{n} = (2, 3, 2, 2)$,

$$\mathbf{r} = \boxed{\begin{array}{cccc} 2 & 2 & 0 & 0 \\ & 3 & 1 & 1 \\ & & 2 & 2 \\ & & & 2 \end{array}} \qquad \mathbf{m} = \boxed{\begin{array}{cccc} 0 & 2 & 0 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 1 \\ & & & 0 \end{array}}$$

where r_{ij} and m_{ij} are written in the usual matrix positions. Note that r_{ij} is obtained by summing the entries in $\mathbf{m}$ weakly above and to the right of m_{ij}.

Then we get $(a_1, a_2, a_3, a_4) = (2, 5, 7, 9)$, $n = 9$, and

$$\tau^{\max} = (89 \subset 12589 \subset 1234589 \subset [9]), \qquad \tau^{\mathbf{r}} = (45 \subset 12459 \subset 1234589 \subset [9]),$$

which correspond to the cosets in $W/W_{\mathbf{n}}$

$$w^{\max} = 89|125|34|67, \qquad w^{\mathbf{r}} = 45|129|38|67.$$

(The minimal-length representatives of these cosets are the permutations as written; the other elements are obtained by permuting numbers within each block.) The partial flag variety is $\mathrm{Fl} = \{U_1 \subset U_2 \subset U_3 \subset \mathbf{k}^9 \mid \dim U_i = a_i\}$, and the Schubert varieties are:

$$X(\tau^{\max}) = \left\{ U_\bullet \,\middle|\, \begin{array}{c} \mathbf{k}^2 \subset U_2 \\ \mathbf{k}^5 \subset U_3 \end{array} \right\}, \quad X(\tau^{\mathbf{r}}) = \left\{ U_\bullet \,\middle|\, \begin{array}{c} U_1 \subset \mathbf{k}^5 \subset U_3,\ \mathbf{k}^2 \subset U_2 \\ \dim U_2 \cap \mathbf{k}^5 \geq 4 \end{array} \right\}.$$

The opposite cells $Y(\tau)$ are defined by the extra conditions $U_i \cap E'_i = 0$.

Example. Fulton's universal degeneracy schemes [8]. Given $m > 0$, let Z be the affine space associated to the quiver data $h = 2m$, $\mathbf{n} = (1, 2, \ldots, m, m, \ldots, 2, 1)$. For each $w \in S_{m+1}$, Fulton defines a "degeneracy scheme" $\Omega_w = Z(\mathbf{r})$ as follows. (Here $\Omega_w = Z(\mathbf{r})$ is a variety. We will define scheme structures for quiver varieties in §2.) Denote $\bar{i} = 2m + 1 - i$, and define $\mathbf{r} = \mathbf{r}(w) = (r_{ij})$ and $\mathbf{m} = (m_{ij})$ by:

$$\begin{array}{c} r_{ij} = r_{\overline{j}\overline{i}} = i \\ r_{i\bar{j}} = \#\,[i] \cap w[j] \end{array} \qquad m_{ij} = \left\{ \begin{array}{ll} 1, & (i,j) = (w(k), \bar{k}),\ \exists k \leq m \\ 0, & \text{otherwise} \end{array} \right.$$

for $1 \leq i, j \leq m$. The associated Schubert varieties $Y(\tau^{\mathbf{r}})$ are given by $\tau^{\mathbf{r}} = (\tau^{\mathbf{r}}_1 \subset \cdots \subset \tau^{\mathbf{r}}_{\bar{1}})$ or by cosets $\widetilde{w} = \widetilde{w}_1 | \cdots | \widetilde{w}_{\bar{1}} \in W/W_{\mathbf{n}}$

$$\begin{array}{c} \tau^{\mathbf{r}}_i = [a_{i-1}] \cup \{a_{\overline{w^{-1}(1)}}, a_{\overline{w^{-1}(2)}}, \ldots, a_{\overline{w^{-1}(i)}}\}, \\ \tau^{\mathbf{r}}_{\bar{i}} = [a_{\bar{i}} - 1] \cup \{a_{\bar{1}}, a_{\bar{2}}, \ldots, a_{\overline{m}}\} \end{array} \qquad \begin{array}{c} \widetilde{w}_i = [a_{i-2} + 1, a_{i-1}] \cup \{a_{\overline{w^{-1}(i)}}\} \\ \widetilde{w}_{\overline{m}} = [a_{m-1}{+}1, a_m{-}1] \cup \{a_{\overline{w^{-1}(m+1)}}\} \\ \widetilde{w}_{\bar{j}} = [a_{\bar{j}-2}{+}1, a_{\bar{j}-1}] \end{array}$$

for $1 \leq i \leq m$, $1 \leq j \leq m{-}1$. Furthermore $\tau^{\max} = \tau^{\mathbf{r}(w)}$ and $\widetilde{w}^{\max} = \widetilde{w}^{\mathbf{r}(w)}$ for $w = e \in S_{m+1}$, the identity permutation.

Example. For a given h and $\mathbf{n}$, the *variety of complexes* is defined as the union $\mathcal{C} = \cup_{\mathbf{r}} Z(\mathbf{r})$ over all $\mathbf{r} = (r_{ij})$ with $r_{i,i+2} = 0$ for each i. The subvarieties $Z(\mathbf{r})$ correspond to the multiplicity matrices $\mathbf{m} = (m_{ij})$ with $m_{ij} = 0$ for all $i + 2 \leq j$, and $m_{ii} + m_{i-1,i} + m_{i,i+1} = n_i$ for all i. Musili-Seshadri [16] find the irreducible components (maximal subvarieties) of $\mathcal{C}$, and show that each component is isomorphic to the opposite cell of some Schubert variety.

Example. The classical *determinantal variety* of $k \times l$ matrices of rank $\leq m$ (where $m \leq k, l$) is $\mathcal{D} = Z(\mathbf{r})$ for $\mathbf{r} = \begin{pmatrix} l & m \\ 0 & k \end{pmatrix}$, and $\mathbf{m} = \begin{pmatrix} l-m & m \\ 0 & k-m \end{pmatrix}$. Also $n = k + l$,

$$\tau^{\max} = ([k+1, n] \subset [n]), \quad \tau^{\mathbf{r}} = ([m+1, l] \cup [n-m+1, n] \subset [n])$$

$$X(\tau^{\max}) = \mathrm{Fl} = \mathrm{Gr}(l, \mathbf{k}^n), \quad X(\tau^{\mathbf{r}}) \cong \{U \in \mathrm{Gr}(l, \mathbf{k}^n) \mid \dim U \cap \mathbf{k}^l \geq l - m\},$$

$$\mathcal{D} = Z(\mathbf{r}) \cong Y(\tau^{\mathbf{r}}) = \{U \in \mathrm{Gr}(l, \mathbf{k}^n) \mid \dim U \cap \mathbf{k}^l \geq l - m,\ U \cap E' = 0\},$$

where $E' = \langle e_{l+1}, e_{l+2}, \ldots, e_n \rangle$.

2 Plucker coordinates and determinantal ideals

In this section we prove the scheme-theoretic version of Zelevinsky's bijection. From now on, we assume our field $\mathbf{k}$ is infinite.

2.1 Coordinates on the opposite big cell

Consider the opposite cell $\mathbf{O} \subset GL(n)/P$. It is easily seen that $\mathbf{O}$ consists of those cosets which have a unique representative A of the form

$$A = (a_{kl}) = \begin{pmatrix} I_1 & 0 & 0 & \cdots & 0 \\ A_{21} & I_2 & 0 & \cdots & 0 \\ A_{31} & A_{32} & I_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ A_{h1} & A_{h2} & A_{h3} & \cdots & I_h \end{pmatrix} \mod P,$$

where I_i is the identity matrix of size n_i, and A_{ij} is an arbitrary matrix of size $n_i \times n_j$. That is, $\mathbf{O}$ is an affine space with coordinates a_{kl} for those positions (k, l) with $1 \leq l \leq a_i < k \leq n$ for some i. Its coordinate ring is the polynomial ring

$$\mathbf{k}[\mathbf{O}] = \mathbf{k}[a_{kl}].$$

For a matrix $M \in M(k \times l)$ and subsets $\lambda \subset [k]$, $\mu \subset [l]$, let $\det M_{\lambda \times \mu}$ be the minor with row indices λ and column indices μ. Now let $\sigma \subset [n]$ be a subset of size $\#\sigma = a_i$ for some i. Define the *Plucker coordinate* $p_\sigma \in \mathbf{k}[\mathbf{O}]$ to be the a_i-minor of our matrix A with row indices σ and column indices the interval $[a_i]$:

$$p_\sigma = p_\sigma(A) = \det A_{\sigma \times [a_i]}.$$

Define a partial order on Plucker coordinates by:

$$\sigma \leq \sigma' \quad \Longleftrightarrow \quad \begin{array}{c} \sigma = \{\sigma(1) < \sigma(2) < \cdots < \sigma(a_i)\}, \\ \sigma' = \{\sigma'(1) < \sigma'(2) < \cdots < \sigma'(a_i)\}, \\ \sigma(1) \leq \sigma'(1),\ \sigma(2) \leq \sigma'(2), \cdots, \sigma(a_i) \leq \sigma'(a_i). \end{array}$$

This is a version of the Bruhat order.

Proposition. *Let $\tau = (\tau_1 \subset \cdots \subset \tau_h = [n])$ be a subset-flag and $Y(\tau)$ the intersection of the Schubert variety $X(\tau)$ with the opposite cell $\mathbf{O}$. Then the (reduced) vanishing ideal $\mathcal{I}(\tau) \subset \mathbf{k}[\mathbf{O}]$ of $Y(\tau) \subset \mathbf{O}$ is generated by those Plucker coordinates p_σ which are incomparable with one of the p_{τ_i}:*

$$\mathcal{I}(\tau) = \langle p_\sigma \mid \exists\, i,\ \#\sigma = a_i,\ \sigma \not\leq \tau_i \rangle.$$

Proof. This follows from well-known results of Lakshmibai-Musili-Seshadri in Standard Monomial Theory (see e.g. [16],[13]).

2.2 The main theorem

Denote a generic element of the quiver space $Z = M(n_2 \times n_1) \times \cdots \times M(n_h \times n_{h-1})$ by $(A_1, \ldots, A_{h-1})$, so that the coordinate ring of Z is the polynomial ring in the entries of all the matrices A_i. Let $\mathbf{r} = (r_{ij})$ index the quiver variety $Z(\mathbf{r}) = \{(A_1, \ldots, A_{h-1}) \mid \operatorname{rank} A_{j-1} \cdots A_i \leq r_{ij}\}$.

Let $\mathcal{J}(\mathbf{r}) \subset \mathbf{k}[Z]$ be the ideal generated by the determinantal conditions implied by the definition of $Z(\mathbf{r})$:

$$\mathcal{J}(\mathbf{r}) = \left\langle \det(A_{j-1}A_{j-2}\cdots A_i)_{\lambda\times\mu} \;\middle|\; \begin{array}{c} j > i,\ \lambda \subset [n_j],\ \mu \subset [n_i] \\ \#\lambda = \#\mu = r_{ij} + 1 \end{array} \right\rangle .$$

Clearly $\mathcal{J}(\mathbf{r})$ defines $Z(\mathbf{r})$ set-theoretically.

Theorem. *$\mathcal{J}(\mathbf{r})$ is a prime ideal and is the vanishing ideal of $Z(\mathbf{r}) \subset Z$. There are isomorphisms of reduced schemes*

$$Z(\mathbf{r}) = \mathit{Spec}(\mathbf{k}[Z] \,/\, \mathcal{J}(\mathbf{r})) \cong \mathit{Spec}(\mathbf{k}[\mathbf{O}] \,/\, \mathcal{I}(\tau^{\mathbf{r}})) = Y(\tau^{\mathbf{r}}).$$

That is, the quiver scheme $Z(\mathbf{r})$ defined by $\mathcal{J}(\mathbf{r})$ is isomorphic to the reduced variety $Y(\tau^{\mathbf{r}})$, the opposite cell of a Schubert variety.

Corollary. *For a ring R, consider the polynomial ring $R[\mathbf{O}] = R[a_{kj}]$ and the ideal $\mathcal{J}(\mathbf{r})_R \subset R[\mathbf{O}]$ generated by the same determinants as above. Define the scheme $Z(\mathbf{r})_R = \mathrm{Spec}(R[\mathbf{O}]/\mathcal{J}(\mathbf{r})_R)$.*
(i) If $\mathbf{k}$ is an arbitrary field, then $Z(\mathbf{r})_{\mathbf{k}}$ is reduced, irreducible, Cohen-Macaulay, normal, and has rational singularities.
(ii) If R is a noetherian ring, then $Z(\mathbf{r})_R$ is reduced (resp. irreducible, Cohen-Macaulay, normal) exactly when $\mathrm{Spec}(R)$ is reduced (irreducible, Cohen-Macaulay,

normal).

Proof of Corollary. (i) Let $\overline{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$. Then the desired properties of $Z(\mathbf{r})_{\overline{\mathbf{k}}}$ follow from the corresponding properties of Schubert varieties (see e.g. [11], [15], [17]). But this implies these properties for $Z(\mathbf{r})_{\mathbf{k}}$ as well, since $Z(\mathbf{r})_{\overline{\mathbf{k}}} \to Z(\mathbf{r})_{\mathbf{k}}$ is a faithfully flat morphism. (See [14], §21.E.)
(ii) By [15], the morphism $Z(\mathbf{r})_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is faithfully flat, so $Z(\mathbf{r})_R \to \mathrm{Spec}(R)$ is as well ([14], §3.C). Now, the fibers of this latter morphism are reduced, Cohen-Macaulay, and normal by (i), so the corresponding properties hold for the total space $Z(\mathbf{r})_R$ exactly when they hold for the base ([14], §21.E). Finally, $Z(\mathbf{r})_R \to \mathrm{Spec}(R)$ is a closed surjective morphism with irreducible fibers of the same dimension, and it is elementary that the total space is irreducible exactly when the base is irreducible (see [18], §I.6.3).

Proof of Theorem. The map of §1.3, $\zeta : Z \xrightarrow{\sim} Y(\tau^{\max}) \subset \mathbf{O}$ is an algebraic isomorphism onto its image, so the restriction homomorphism $\zeta^* : \mathbf{k}[\mathbf{O}] \to \mathbf{k}[Z]$ is surjective.

Now, ζ maps $Z(\mathbf{r})$ isomorphically onto $Y(\tau^{\mathbf{r}})$, so we have

$$\begin{aligned} Z(\mathbf{r}) = \mathrm{Spec}(\mathbf{k}[Z] \,/\, \widetilde{\mathcal{J}}(\mathbf{r})) \quad &\cong \quad \mathrm{Spec}(\mathbf{k}[\mathbf{O}] \,/\, (\zeta^*)^{-1}\widetilde{\mathcal{J}}(\mathbf{r})) \\ &= \quad \mathrm{Spec}(\mathbf{k}[\mathbf{O}] \,/\, \mathcal{I}(\tau^{\mathbf{r}})) \ = \ Y(\tau^{\mathbf{r}}). \end{aligned}$$

where $\widetilde{\mathcal{J}}(\mathbf{r}) \subset \mathbf{k}[Z]$ denotes the (reduced) vanishing ideal of $Y(\tau^{\mathbf{r}})$.

We must show $\mathcal{J}(\mathbf{r}) = \widetilde{\mathcal{J}}(\mathbf{r})$. Since clearly $\mathcal{J}(\mathbf{r}) \subset \widetilde{\mathcal{J}}(\mathbf{r})$, we are left with the other inclusion, which is equivalent to

$$(\zeta^*)^{-1}\mathcal{J}(\mathbf{r}) \supset (\zeta^*)^{-1}\widetilde{\mathcal{J}}(\mathbf{r}) = \mathcal{I}(\tau^{\mathbf{r}}).$$

We prove this in the next section.

2.3 Proof of the main theorem

Let $A \in \mathbf{O}$ be the generic matrix of §2.1. We define ideals $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \subset \mathbf{k}[\mathbf{O}]$ generated by certain minors of A:

$$\begin{aligned} &\mathcal{I}_0 = (\zeta^*)^{-1}\mathcal{J}(\mathbf{r}) \\ &= (\zeta^*)^{-1}\left\langle \det(A_{j-1}A_{j-2}\cdots A_i)_{\lambda\times\mu} \;\middle|\; \begin{array}{c} j > i,\ \lambda \subset [n_j],\ \mu \subset [n_i] \\ \#\lambda = \#\mu = r_{ij}+1 \end{array} \right\rangle . \end{aligned}$$

$$\mathcal{I}_1 \ = \left\langle \det A_{\lambda\times\mu} \;\middle|\; \begin{array}{c} i \le j,\quad \lambda \subset [a_j{+}1, n],\quad \mu \subset [a_i] \\ \#\lambda = \#\mu = r_{ij}{+}1 \end{array} \right\rangle$$

$$\mathcal{I}_2 \ = \ \mathcal{I}(\tau^{\mathbf{r}}) \ = \ \left\langle \det A_{\sigma\times[a_i]} \;\middle|\; \begin{array}{c} 1 \le i \le h{-}1,\quad \sigma \subset [n] \\ \#\sigma = a_i,\quad \sigma \not\le \tau_i^{\mathbf{r}} \end{array} \right\rangle$$

To finish the proof of Theorem 2.2, we will show

$$\mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \ .$$

Lemma 1. *Let $X = (x_{ij})$ and $Y = (y_{kl})$ be matrices of variables x_{ij}, y_{kl} generating a polynomial ring. Let $\mathcal{J}_X$ (resp. $\mathcal{J}_Y$) be the ideal generated by all $r{+}1$-minors of X (resp. Y). Then $\mathcal{J}_X$ and $\mathcal{J}_Y$ both contain all $r{+}1$-minors of the product XY.*

Proof.
$$\det(XY)_{\lambda\times\mu} = \sum_{\nu} \det X_{\lambda\times\nu}\, \det Y_{\nu\times\mu}. \quad \bullet$$

Lemma 2. *Let $(A_1, \ldots, A_{h-1})$ be a generic element of Z, and for $j > i$ let $\mathcal{J}_{ji}$ be the ideal generated by all $r+1$-minors of the $n_j \times n_i$ product matrix $A_{j-1} \cdots A_i$. Then $\mathcal{J}_{ji}$ contains all $r{+}1$-minors of the $(n{-}a_{j-1}) \times a_i$ matrix*

$$\widetilde{A}^{(ji)} = \begin{pmatrix} A_{j-1}\cdots A_1 & A_{j-1}\cdots A_2 & \cdots & A_{j-1}\cdots A_i \\ A_j\cdots A_1 & A_j\cdots A_2 & \cdots & A_j\cdots A_i \\ \vdots & \vdots & & \vdots \\ A_{h-1}\cdots A_1 & A_{h-1}\cdots A_2 & \cdots & A_{h-1}\cdots A_i \end{pmatrix}$$

Proof. Note that we can factor the matrix

$$\widetilde{A}^{(ji)} = \begin{pmatrix} I_{j-1} \\ A_j \\ \vdots \\ A_{h-1}\cdots A_j \end{pmatrix} \cdot\ A_{j-1}\cdots A_i\ \cdot\ (A_{i-1}\cdots A_1,\ A_{i-1}\cdots A_2,\ \cdots\ ,\ A_{i-1},\ I_i).$$

Now apply Lemma 1 twice. $\bullet$

Lemma 3. $\quad \mathcal{I}_0 \supset \mathcal{I}_1$.

Proof. For generic elements $A \in \mathbf{O}$ and $(A_1, \ldots, A_{h-1}) \in Z$, we have by definition $\zeta^*(f(A)) = f(\zeta(A_1, \ldots, A_{h-1}))$ for any polynomial f in the matrix entries. Now let $\lambda \subset [a_{j-1}{+}1, n]$, $\mu \subset [a_i]$, $\#\lambda = \#\mu = r_{ij} + 1$, and consider a generator $\det A_{\lambda\times\mu}$ of $\mathcal{I}_1$. Then

$$\zeta^*(\det A_{\lambda\times\mu}) = \det \zeta(A_1, \ldots, A_{h-1})_{\lambda\times\mu} = \det \widetilde{A}^{(ji)}_{\lambda'\times\mu}$$

where $\lambda' \subset [n - a_{j-1}]$ is a translate of λ. By Lemma 2, $\det \widetilde{A}^{(ji)}_{\lambda'\times\mu} \in \mathcal{J}(\mathbf{r})$, so $\mathcal{I}_1 = \langle \det A_{\lambda\times\mu} \rangle \subset (\zeta^*)^{-1}\mathcal{J}(\mathbf{r}) = \mathcal{I}_0$. $\bullet$

Lemma 4. *(Gonciulea-Lakshmibai) Let A be a generic element of $\mathbf{O}$. Let $1 \le t \le a_i$, $1 \le s \le n$, and $\sigma = \{\sigma(1) < \sigma(2) < \cdots < \sigma(a_i)\} \subset [n]$ with $\sigma(a_i - t + 1) \ge s$. Then $p_\sigma(A)$ belongs to the ideal of $\mathbf{k}[\mathbf{O}]$ generated by t-minors of A with row indices $\ge s$ and column indices $\le a_i$.*

Proof. Choose $\sigma' \subset [s, n] \cap \sigma$ with $\#\sigma' = t$, and let $\sigma'' = \sigma \backslash \sigma'$. Then the Laplace expansion of $p_\sigma(A)$ with respect to the rows σ', σ'', gives

$$p_\sigma(A) = \det A_{\sigma\times[a_i]} = \sum_{\lambda'\cup\lambda''=[a_i]} \pm \det A_{\sigma'\times\lambda'} \det A_{\sigma''\times\lambda''},$$

where the sum is over all partitions of the interval $[a_i]$. The first factor of each term in the sum is of the form required. $\bullet$

Lemma 5. $\mathcal{I}_1 \supset \mathcal{I}_2$.

Proof. Let $\sigma \subset [n]$ with $\#\sigma = a_i$, $\sigma \not\leq \tau_i^{\mathbf{r}}$ for some i, $1 \leq i \leq h-1$. Now, $\tau_i^{\mathbf{r}}$ has the largest possible entries such that

$$\tau_i^{\mathbf{r}}(a_i - r_{i,j+1}) \leq a_j, \qquad \forall\, j \geq i,$$

so $\sigma \not\leq \tau_i^{\mathbf{r}}$ must violate this condition for some j:

$$\sigma(a_i - r_{i,j+1}) \geq a_j + 1, \qquad \exists\, j \geq i.$$

Hence by Lemma 4, $p_\sigma(A)$ is in $\mathcal{I}_1$. $\bullet$

The Main Theorem 2.2 is therefore proved.

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