Let r_n be the number of rooted, unlabelled trees with n vertices: $r_0 = 0$, $r_1 = r_2 = 1$, $r_3 = 2$, $r_4 = 4$. For $n = 4$, the tree can be either: a linear path rooted at an end or an internal vertex; or a star rooted at the center or at a leaf; giving $r_4 = 4$ distinct choices.

The combinatorial specification $\mathcal{R} = [1] \times \mathrm{MS{\scriptsize ET}}(\mathcal{R})$ gives the following equation involving the generating function $R(x) = \sum_{n\geq 1} r_n x^n$:

$$R(x) = x \prod_{j\geq 1} \frac{1}{(1-x^j)^{r_j}}\,.$$

We will apply logarithmic differentiation to obtain an amazing recurrence for r_n. Writing out the equation as:

$$\sum_{n\geq 0} r_{n+1}x^n = \prod_{j\geq 1}(1-x^j)^{-r_j}\,,$$

we apply the operation $x\frac{d}{dx}\log$ to both sides. On the left side, the identity $x\frac{d}{dx}\log f(x) = xf'(x)/f(x)$ implies:

$$x\frac{d}{dx}\log\sum_{n\geq 0} r_{n+1}x^n = \frac{\sum_{n\geq 1} n r_{n+1}x^n}{\sum_{m\geq 0} r_{m+1}x^m}\,.$$

On the right side, we use $\log(ab) = \log a + \log b$ and $\log(a^b) = b\log a$ to get:

$$x\frac{d}{dx}\log\prod_{j\geq 1}(1-x^j)^{-r_j} = \sum_{j\geq 1} -r_j\, x\frac{d}{dx}\log(1-x^j)$$

$$= \sum_{j\geq 1} r_j \frac{jx^j}{1-x^j} = \sum_{j\geq 1}\sum_{i\geq 1} jr_j\, x^{ij} = \sum_{k\geq 1}(\sum_{j|k} jr_j)x^k\,.$$

where in the last equality we substitute $k = ij$, and $j|k$ means j divides k.

Now equating the two sides, clearing the denominator, and collecting x^n terms, we get:

$$\sum_{n\geq 1} nr_{n+1}x^n = \sum_{k\geq 1}(\sum_{j|k} jr_j)x^k \cdot \sum_{m\geq 0} r_{m+1}x^m = \sum_{n\geq 1}(\sum_{k=1}^{n}\sum_{j|k} jr_j r_{n-k+1})x^n$$

where in the second equality we substitute $n = k+m$, so that $m+1 = n-k+1$. Thus:

$$r_{n+1} = \frac{1}{n}\sum_{k=1}^{n}\sum_{j|k} jr_j r_{n-k+1}\,,$$

where the right side involves only $r_1, \ldots, r_n$. This recurrence has no combinatorial explanation, but it is fairly efficient computationally.

EXAMPLE: To compute r_5, we sum over $k = 1, 2, 3, 4$ and j running over all divisors of k: that is, $(j,k) = (1,1)$, (1,2), (2,2), (1,3), (3,3), (1,4), (2,4), (4,4), so that:

$$\begin{aligned} r_5 &= \tfrac{1}{4}(r_1r_4 + r_1r_3 + 2r_2r_3 + r_1r_2 + 3r_3r_2 + r_1r_1 + 2r_2r_1 + 4r_4r_1) \\ &= \tfrac{1}{4}(4+2+4+1+6+1+2+16) = 9\,. \end{aligned}$$