Inversion of Formal Power Series. We extend the ring of formal power series $\mathbb{C}[[x]]$ to the field of formal Laurent series $\mathbb{C}((x))$:

$$\mathbb{C}((x)) = \left\{ \textstyle\sum_{k\geq -N} a_k x^k \;\middle|\; N \in \mathbb{Z}\,, a_k \in \mathbb{C} \right\}.$$

These are the series in x, x^{-1} with a lowest term x^{-N}, but not necessarily a highest term. We define the operator $[x^n]$ which extracts the x^n coefficient of a series: $[x^n]\left(\sum_k a_k x^k\right) = a_n$.

LEMMA: (i) For $h(x) \in \mathbb{C}((x))$, we have $[x^{-1}]h'(x) = 0$.
(ii) For $f(x) \in x\mathbb{C}[[x]]$ with $[x^1]f(x) \neq 0$, and $i \in \mathbb{Z}$, we have:

$$[x^{-1}]f(x)^i f'(x) = \begin{cases} 1 & \text{if } i = -1 \\ 0 & \text{else.} \end{cases}$$

Proof. (i) Obvious from the definition of derivative: $(x^k)' = kx^{k-1}$ for $k \in \mathbb{Z}$.
(ii) For $i \neq -1$, this follows from (i), since $f(x)^i f'(x) = \frac{1}{i+1}(f(x)^{i+1})'$. For $i = -1$ and $f(x) = \sum_{k\geq 1} a_k x^k$:

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{a_1 + 2a_2 x + \cdots}{a_1 x + a_2 x^2 + + \cdots} = \frac{a_1 + 2a_2 x + \cdots}{a_1 x} \cdot \frac{1}{1 + \left(\frac{2a_2}{a_1}x + \frac{3a_3}{a_1}x^2 + \cdots\right)} \\ &= \left(x^{-1} + \tfrac{2a_2}{a_1} + \tfrac{3a_3}{a_1}x + \cdots\right)\left(1 - x\left(\tfrac{2a_2}{a_1} + \tfrac{3a_3}{a_1}x + \cdots\right) + \cdots\right), \end{aligned}$$

from which $[x^{-1}]f(x)^{-1}f'(x) = 1$ is evident. $\square$

LAGRANGE INVERSION THEOREM: Let $f(x), g(x) \in x\mathbb{C}[[x]]$ be inverses: $f(g(x)) = x$. Then:

$$[x^n]g(x) = \frac{1}{n}[x^{-1}]\frac{1}{f(x)^n}.$$

In particular, if $f(x) = x/\phi(x)$ and $g(x) = x\,\phi(g(x))$, then:

$$[x^n]g(x) = \frac{1}{n}[x^{n-1}]\phi(x)^n.$$

Proof. Let $g(x) = \sum_{i\geq 1} b_i x^i$. Since $f = g^{-1}$, we have:

$$x = g(f(x)) = \sum_{i\geq 1} b_i f(x)^i,$$

and taking the derivative gives:

$$1 = \sum_{i\geq 1} i b_i \left(f(x)^i\right)' = \sum_{i\geq 1} i b_i f(x)^{i-1} f'(x).$$

We wish to move the b_n term to be the coefficient of $f(x)^{-1}f'(x)$. Thus, we divide by $f(x)^n$:

$$\begin{aligned}\frac{1}{f(x)^n} &= \sum_{i\geq 1} i b_i f(x)^{i-1-n} f'(x) \\ &= \sum_{i=1}^{n-1} \frac{ib_i}{i-n}\left(f(x)^{i-n}\right)' + nb_n\frac{f'(x)}{f(x)} + \sum_{i>n}\frac{ib_i}{i-n}\left(f(x)^{i-n}\right)'\end{aligned}$$

Applying the Lemma to each term, we have the first formula: $[x^{-1}][1/f(x)^n] = nb_n$.

For the second formula, take $f(x) = x/\phi(x)$ so that $x = f(g(x)) = g(x)/\phi(g(x))$ is equivalent to $g(x) = x\,\phi(g(x))$. Now, evidently $[x^{-1}]h(x) = [x^{n-1}](x^n h(x))$, so:

$$b_n = \frac{1}{n}[x^{-1}]\frac{1}{f(x)^n} = \frac{1}{n}[x^{n-1}]\frac{x^n}{x^n/\phi(x)^n} = \frac{1}{n}[x^{n-1}]\phi(x)^n. \qquad \square$$

Reference: Richard Stanley, *Enumerative Combinatorics*, Vol. 2, Ch. 5.

Inversion of Analytic Functions. We give an analytic proof of Lagrange Inversion. Consider a function $f(u)$ of a complex variable u, holomorphic in a neighborhood of $u = 0$. Suppose $f(0) = 0$ and $f'(0) \neq 0$, so by the Inverse Function Theorem, $f(u)$ is one-to-one inside a small circle $\mathcal{C}$ defined by $|u| = \epsilon$, and there is a unique inverse function $g(z)$ defined near $z = 0$ with $g(f(u)) = u$. Applying the Cauchy Residue Theorem and then a change of variables $u = g(\zeta)$, $\zeta = f(u)$, $d\zeta = f'(u)\,du$, we have:[1]

$$g(z) \;=\; \frac{1}{2\pi i}\oint_{f(\mathcal{C})}\frac{g(\zeta)}{\zeta - z}\,d\zeta \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{f'(u)}{f(u)-z}\,u\,du.$$

Expanding $\frac{f'(u)}{f(u)-z} = \frac{f'(u)}{f(u)}\frac{1}{1-z/f(u)} = \frac{f'(u)}{f(u)}\sum_{n\geq 0}(\frac{z}{f(u)})^n$, we get the Taylor series:

$$g(z) \;=\; \sum_{n\geq 0}\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{f'(u)}{f(u)}\frac{z^n}{f(u)^n}\,u\,du \;=\; \sum_{n\geq 0} b_n z^n$$

where:

$$b_n \;=\; [z^n]g(z) \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{f'(u)}{f(u)^{n+1}}\,u\,du \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{1}{n}\,\frac{1}{f(u)^n}\,du \;=\; \frac{1}{n}\,[u^{-1}]\frac{1}{f(u)^n}.$$

Here the third equality is integration by parts, and the fourth is the Residue Theorem.

Generalization: For inverse functions with $g(f(x)) = x$, we can use the same reasoning to expand $h(g(x))$ for any $h(x)$ with $h(0) = 0$, obtaining:

$$[x^n]h(g(x)) = \frac{1}{n}\,[x^{-1}]\frac{h'(x)}{f(x)^n}.$$

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[1] For a function defined by a Laurent series $g(\zeta) = \sum_{n\in\mathbb{Z}} b_n(\zeta - z)^n$ which is holomorphic on the disk D defined by $|\zeta - z| \leq \delta$, the Cauchy residue is: $g(z) = b_0 = \frac{1}{2\pi i}\oint_{|\zeta - z| = \delta}\frac{g(\zeta)}{\zeta - z}\,d\zeta$.

Now let $f(u)$ be holomorphic on a simply connected region $\Omega \subset \mathbb{C}$ bounded by a simple closed curve $\mathcal{C}$, with $f(\Omega) \subset D$. Let $u = u_1, \ldots, u_N \in \Omega$ be the solutions of $f(u) = z$, where u_i has multiplicity m_i. Then $\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{f'(u)}{f(u)-z}\,du = \sum_{i=1}^N m_i$ counts solutions. For $h(u)$ holomorphic, $\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{f'(u)}{f(u)-z}\,h(u)\,du = \sum_{i=1}^N m_i h(u_i)$.