

Feel free to discuss homework problems with other students, and to learn from references, but please do not look up specific answers. Write out solutions in your own words, and give explicit credit for any significant help.

1. Show that the number e_{2k} of permutations in \mathfrak{S}_{2k} which have only even-length cycles is $(1 \cdot 3 \cdot 5 \cdots (2k-1))^2$. For example, $e_4 = (1 \cdot 3)^2 = 9$ counts the permutations:

$$(1234), (1243), (1324), (1342), (1423), (1432), (12)(34), (13)(24), (14)(23).$$

Hint: The even part of a function $f(x)$ is $\frac{1}{2}(f(x) + f(-x))$.

Extra Credit: Give a bijective transformation proof of this formula.

2. Writing permutations as sets of cycles, we computed the bivariate generating function:

$$\tilde{S}(x, y) = \sum_{k, n \geq 0} \binom{k}{n} y^n \frac{x^k}{k!} = \exp(y \log \frac{1}{1-x}) = \frac{1}{(1-x)^y},$$

Consider the average or expected number of cycles in a permutation $w \in \mathfrak{S}_k$:

$$a_k = \frac{1}{k!} \sum_{w \in \mathfrak{S}_k} \text{cyc}(w) = \frac{1}{k!} \sum_{n=1}^k \binom{k}{n} n.$$

Show that the generating function of a_k is a partial derivative of $\tilde{S}(x, y)$:

$$A(x) = \sum_{k \geq 0} a_k x^k = \left[\frac{\partial}{\partial y} \tilde{S}(x, y) \right]_{y=1}.$$

Compute $A(x)$, and find a simple asymptotic approximation $a_k \sim f(k)$: what is the approximate number of cycles expected for a permutation $w \in \mathfrak{S}_k$? *Hint:* Harmonic series.

Burnside Lemma: For G acting on a finite class \mathcal{F} , the number of orbits is equal to the average number of fixed points: $\#\bar{\mathcal{F}} = \frac{1}{\#G} \sum_{g \in G} \#\mathcal{F}^g$, where $\mathcal{F}^g = \{f \in \mathcal{F} \mid g(f) = f\}$.

Pattern inventory generating function: For fixed k, n , and the finite class of functions $\mathcal{F} = \{f : [k] \rightarrow [n]\}$, define the multi-variable generating function:

$$F(\vec{y}) = F(y_1, \dots, y_n) = \sum_{f \in \mathcal{F}} y_1^{\text{wt}_1(f)} \cdots y_n^{\text{wt}_n(f)} = \sum_{m_1, \dots, m_n \geq 0} F_{m_1, \dots, m_n} y_1^{m_1} \cdots y_n^{m_n},$$

where $\text{wt}_i(f) = \#f^{-1}(i)$ and $F_{\vec{m}} = F_{m_1, \dots, m_n} = \#\{f \mid \text{wt}_1(f) = m_1, \dots, \text{wt}_n(f) = m_n\}$.

Cycle index polynomial: For a permutation group $G \subset \mathfrak{S}_k$ define:

$$Z_G(z_1, \dots, z_k) = \sum_{w \in G} z_1^{\text{cyc}_1(w)} \cdots z_k^{\text{cyc}_k(w)} = \sum_{c_1, \dots, c_k \geq 0} G_{c_1, \dots, c_k} z_1^{c_1} \cdots z_k^{c_k},$$

where $\text{cyc}_i(w) = \# i$ -cycles in $w \in \mathfrak{S}_k$ and $G_{c_1, \dots, c_k} = \#\{w \in G \mid \text{cyc}_i(w) = c_i \text{ for all } i\}$.

3. *Polya formula:* Let $G \subset \mathfrak{S}_k$ act on $\mathcal{F} = \{f : [k] \rightarrow [n]\}$, with orbits $\bar{\mathcal{F}} = \mathcal{F}/G$. Then:

$$\bar{F}(\vec{y}) = \frac{1}{\#G} Z_G(p_1(\vec{y}), \dots, p_k(\vec{y})),$$

where $p_i(\vec{y}) = y_1^i + \dots + y_n^i$. Prove this as follows:

a. Show $\bar{F}(\vec{y}) = \frac{1}{\#G} S(\vec{y})$, where $S(\vec{y})$ is the multi-variable generating function of:

$$\mathcal{S} = \{(w, f) \in G \times \mathcal{F} \mid f(w(i)) = f(i) \text{ for all } i\},$$

with $\text{wt}_i(w, f) = \text{wt}_i(f) = \#f^{-1}(i)$. *Hint:* Apply Burnside to each weight set $\mathcal{F}_{\vec{m}}$.

b. For the multi-graded class $\mathcal{N} = [n]$ with pattern inventory $N(\vec{y}) = y_1 + \dots + y_n$, give a multi-graded bijection:

$$\mathcal{S} \cong \prod_{w \in G} (\Delta^1 \mathcal{N})^{\text{cyc}_1(w)} \times \dots \times (\Delta^k \mathcal{N})^{\text{cyc}_k(w)},$$

where $\Delta^i \mathcal{N}$ is the diagonal class $\{(j, \dots, j) \in \mathcal{N}^i\}$. Use this to prove Polya's formula.

4. Let G be the rotational symmetry group of the 3-dimensional cube, containing $\frac{1}{4}$ & $\frac{1}{2}$ rotations around opposite pairs of square faces, $\frac{1}{3}$ rotations around opposite pairs of vertices, and $\frac{1}{2}$ rotations of opposite pairs of edges. (You may rely on pictures, no need to codify the representation in matrices.)

a. Show G is isomorphic to \mathfrak{S}_4 . *Hint:* Find 4 isometric structures inside the cube, permuted by G , so that certain rotations induce the generators $(12), (23), (34) \in \mathfrak{S}_4$.

b. Count the possible designs for dice, ways to label the 6 square faces with every element of $1, \dots, 6$, up to rotational symmetry: that is, count the physically distinct designs.

c. Consider the permutation representation $G \subset \mathfrak{S}_6$ induced by the rotations permuting the 6 faces. Determine the cycle index polynomial $Z_G(z_1, \dots, z_6)$.

d. Let $\mathcal{F} = \{f : [6] \rightarrow [n]\}$ be the class of labelings of the 6 faces with n available colors. Use Burnside to compute the number of physically distinct labelings in $\bar{\mathcal{F}} = \mathcal{F}/G$.

e. Use Polya to determine the pattern inventory polynomial $\bar{F}(y_1, \dots, y_n)$. Expand this by computer for $n = 3$, and find how many colorings of the 6 faces use the labels $1, 1, 2, 2, 3, 3$.

5. Define the *Euler phi-function* $\varphi(k) = \#\{j \leq k \mid \gcd(j, k) = 1\}$, the number of integers in $[k]$ which are relatively prime to k . For example, $\varphi(p) = p-1$ for p prime.

a. Use the Principle of Inclusion-Exclusion to prove that if k has prime factorization $k = p_1^{\ell_1} p_2^{\ell_2} \dots$, then $\varphi(k) = \prod_i p_i^{\ell_i-1} (p_i-1)$.

b. Determine the cycle index $Z_G(z_1, \dots, z_k)$ for the cyclic group $G = C_k$ generated by $(12 \dots k) \in \mathfrak{S}_k$. The answer will involve the phi-function.

c. For a graded unlabeled class \mathcal{B} , consider the class $\bar{\mathcal{F}} = \text{CYC}_k \mathcal{B}$ consisting of all cyclic arrangements of k elements of \mathcal{B} , up to rotation symmetry. Use the previous problems to find the generating function $\bar{F}(y)$ in terms of the ordinary generating function $B(y)$.