

Feel free to discuss homework problems with other students, and to learn from references, but please do not look up specific answers. Write out solutions in your own words, and give explicit credit for any significant help.

Recall constructions for labeled $\tilde{\mathcal{C}}$ with exponential generating function $\tilde{C}(x) = \sum_{k \geq 0} C_k \frac{x^k}{k!}$.

$$\begin{array}{ll} \tilde{\mathcal{C}} = \tilde{\mathcal{A}} * \tilde{\mathcal{B}}, & \tilde{C}(x) = \tilde{A}(x)\tilde{B}(x) & \tilde{\mathcal{C}} = \tilde{\mathcal{A}}^{\min} * \tilde{\mathcal{B}}, & \tilde{C}(x) = \int \tilde{A}'(x)\tilde{B}(x) dx \\ \tilde{\mathcal{C}} = \text{SEQ}_n \tilde{\mathcal{A}}, & \tilde{C}(x) = \tilde{A}(x)^n & \tilde{\mathcal{C}} = \text{SEQ} \tilde{\mathcal{A}}, & \tilde{C}(x) = 1/(1-\tilde{A}(x)) \\ \tilde{\mathcal{C}} = \text{SET}_n \tilde{\mathcal{A}}, & \tilde{C}(x) = \tilde{A}(x)^n/n! & \tilde{\mathcal{C}} = \text{SET} \tilde{\mathcal{A}}, & \tilde{C}(x) = \exp \tilde{A}(x) \\ \tilde{\mathcal{C}} = \text{CYC}_n \tilde{\mathcal{A}}, & \tilde{C}(x) = \tilde{A}(x)^n/n & \tilde{\mathcal{C}} = \text{CYC} \tilde{\mathcal{A}}, & \tilde{C}(x) = -\log(1-\tilde{A}(x)) \end{array}$$

There are no labeled multisets: distinct labels prevent exact copies within an object.

1. The class of *derangements* $\tilde{\mathcal{D}}_k$ comprises all permutations $w \in \mathfrak{S}_k$ with $w(i) \neq i$ for all i . Analyze $\tilde{\mathcal{D}}$ using the above constructions, find the generating function $\tilde{D}(x)$, and obtain a formula for D_k . (*Hint*: A permutation is a set of cycles.) Compare this formula with the P.I.E. formula from the previous HW.

2. Recall the Euler number E_k which counts the down-up permutations $\pi \in \mathfrak{S}_k$ with $\pi(1) > \pi(2) < \pi(3) > \dots$. For example, $E_3 = 2$ counts the permutations $\pi = 213, 312$. We split these into the permutations $\tilde{\mathcal{J}}$ with odd length $k = 2\ell + 1$, so that $J_{2\ell+1} = E_{2\ell+1}$ and $J_{2\ell} = 0$; and also the permutations $\tilde{\mathcal{K}}$ with even length $k = 2\ell$, so that $K_{2\ell} = E_{2\ell}$ and $K_{2\ell+1} = 0$.

Deleting $\pi(j) = 1$ from an odd permutation produced the recurrence:

$$\tilde{\mathcal{J}} \cong \tilde{\mathcal{J}} \star [1]^{\min} \star \tilde{\mathcal{J}} + [1],$$

which can be solved via generating functions to give $\tilde{J}(x) = \tan(x)$.

a. Use a similar method to determine $\tilde{K}(x)$.

b. Check that the smallest complex singularities of $\tilde{E}(x) = \tilde{J}(x) + \tilde{K}(x)$ are $\alpha = \frac{\pi}{2}, \beta = -\frac{\pi}{2}$. Find the residues A, B such that:

$$\tilde{E}(x) = \frac{A}{1-x/\alpha} + \frac{B}{1-x/\beta} + f(x),$$

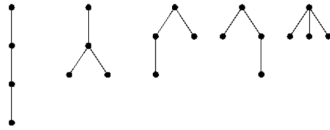
where $f(x)$ is analytic (non-singular) on $|x| \leq \frac{3}{2}\pi$. Hints: In the complex plane $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$. The residue of $\tilde{E}(x)$ at $x = \alpha$ is computed by $A = \lim_{x \rightarrow \alpha} \tilde{E}(x)(1-x/\alpha)$.

By Flajolet-Sedgewick Thm IV.10 p. 258, this implies the asymptotic expansion

$$\frac{E_k}{k!} \sim A\alpha^{-k} + B\beta^{-k}.$$

Use Stirling's approximation $k! \sim \sqrt{2\pi k} (k/e)^k$ to get a simple asymptotic for E_k . Look up E_{99} and E_{100} (in a standard Taylor series) to see how accurate this is in percentage terms.

3. Tree Structures. A *plane tree* is a combinatorial graph with k unlabeled vertices: an ancestor (root vertex) at the top, and below each vertex is a list of children in age order. Let \mathcal{B}_k be the class of such trees, B_k the counting number. E.g. $B_4 = 5$:



The difference between the 3rd and 4th trees is whether the ancestor's older or younger child has a child. Set $\mathcal{B}_0 = \{\}$. Deleting the root gives recursively:

$$\mathcal{B} \cong \{\bullet\} \times \text{SEQ}(\mathcal{B}),$$

where $\{\bullet\}$ denotes the class with a single element of size 1, and

$$\text{SEQ}(\mathcal{B}) = \{\emptyset\} \sqcup \mathcal{B} \sqcup \mathcal{B} \times \mathcal{B} \sqcup \dots$$

denotes all lists of elements in \mathcal{B} . By the Graded Product Principle:

$$B(x) = x(1 + B(x) + B(x)^2 + \dots) = \frac{x}{1 - B(x)}.$$

Solving gives $B(x) = \frac{1}{2}(1 - \sqrt{1 - 4x})$; comparing this to the Catalan generating function implies $B_k = C_{k-1} = \frac{1}{k} \binom{2k-2}{k-1}$.

The Catalan formula can be obtained easily by Lagrange inversion: if a power series $f(x)$ has inverse function $g(x)$, so that $f(g(x)) = x$, then the x^k coefficient of $g(x) = f^{-1}(x)$ is given by: $\frac{1}{k} [x^{-1}] \frac{1}{f(x)^k}$. Here $B(x)(1 - B(x)) = x$, so $f(x) = x(1 - x)$.

PROBLEM: Use the above techniques to count the following types of rooted trees:

- a. Binary plane trees: k unlabeled vertices, no children or two ordered children.
- b. Labeled plane trees: k labeled vertices, any number of ordered children.
- c. Increasing trees: k labeled vertices, any number of *unordered* children, and each child must have a larger label than its parent (so the root must be 1). For example, $k = 2$ gives two trees: $1-2-3$ and $1-\{2,3\}$. *Hint:* Use the directed labeled product $\mathcal{A}^{\min} * \mathcal{B}$. *Extra Credit:* Bijectively prove the simple formula that results.
- d. Ternary trees: k unlabeled vertices with the root at top, and each vertex having zero or three ordered children. *Hint:* Any ternary tree has $k = 3j + 1$ vertices.

4. Formal power series and connected graphs. We define the ring $\mathbb{C}[[x]]$ as the set of all infinite sequences of complex numbers (a_0, a_1, a_2, \dots) , which we write in the notation of power series $A(x) = a_0 + a_1x + a_2x^2 + \dots$, even when the series does not converge for any $x \neq 0$. Addition and multiplication are identical to the corresponding operations on power series: $(\sum_i a_i x^i)(\sum_j b_j x^j) = \sum_k c_k x^k$ with $c_k = \sum_{i+j=k} a_i b_j$; the additive identity is $0 = 0 + 0x + 0x^2 + \dots$ and the multiplicative identity $1 = 1 + 0x + 0x^2 + \dots$.

The ring of complex analytic functions near $x = 0$ embeds as a subring of $\mathbb{C}[[x]]$, by mapping a function to its (convergent) Taylor series. However, $\mathbb{C}[[x]]$ includes elements like the (divergent) generating function of all labeled graphs, $\tilde{G}(x) = \sum_{k \geq 0} 2^{\binom{k}{2}} \frac{x^k}{k!}$. Any formula which holds for analytic functions is guaranteed to hold for all series in $\mathbb{C}[[x]]$, provided it involves only finitely many terms in the coefficient of each x^k . For example, multiplication of power series is associative because the multiplication of analytic functions is so.

a. For a list of formal series $A^{(n)}(x) = \sum_k a_k^{(n)} x^k$, define the *formal limit* $\lim_{n \rightarrow \infty} A^{(n)}(x) = A(x)$ to mean: for each k , there is some $n(k)$ such that $a_k^{(n)} = a_k$ for all $n \geq n(k)$; i.e., the k^{th} coefficient stabilizes for large enough n . (Also called *convergence in the x -adic topology*.)

For $A(x), B(x) \in \mathbb{C}[[x]]$, we define the composite series as:

$$A(B(x)) = \lim_{n \rightarrow \infty} a_0 + a_1 B(x) + a_2 B(x)^2 + \dots + a_n B(x)^n + \dots$$

PROBLEM: Show that this limit formally converges if and only if $b_0 = 0$ or $A(x)$ is a polynomial (of finite degree).

b. Since a graph is a set of connected subgraphs, the Exponential Formula implies $\tilde{G}(x) = \exp \tilde{C}(x)$, where $\tilde{C}(x)$ is the exponential generating function of connected graphs on vertices labeled $\{1, \dots, k\}$. Using the fact that an analytic inverse function is also a formal inverse, this implies:

$$\tilde{C}(x) = \sum_{k \geq 1} c_k \frac{x^k}{k!} = \log \tilde{G}(x) = \log \left(1 + \sum_{k \geq 1} 2^{\binom{k}{2}} \frac{x^k}{k!} \right),$$

where the outside function of the composition is the series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$. (I use small c_k to avoid confusion with Catalan numbers.)

Find a way to evaluate this series in Wolfram Alpha or Mathematica, at least far enough to compute c_k for $k = 4$. Verify the result by directly enumerating the connected graphs on 4 vertices, which can have 3,4,5, or 6 edges: count labeled graphs per unlabeled graph.

c. Use the *Dlog Method* to find a recurrence for C_k . That is, given $e^{\tilde{C}(x)} = \tilde{G}(x)$, perform logarithmic differentiation $\frac{d}{dx} \log$ to both sides, clear denominators, and equate coefficients on the two sides. Finally, solve for c_{k+1} .