Feel free to discuss homework problems with other students, and to learn from references, but please do not look up specific answers. Write out solutions in your own words, and give explicit credit for any significant help. LaTeX is encouraged, but not required.

Let $\mathcal{P}$ be the class of partitions $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_\ell > 0\}$ with size $|\lambda| = \lambda_1 + \cdots \lambda_\ell = k$ and length $\ell(\lambda) = \ell$, counted by $p(k) = \#\mathcal{P}_k$. In terms of multiplicities, $\lambda = 1^{m_1}2^{m_2}\cdots$, where $m_j = \#\{i \text{ with } \lambda_i = j\}$. The Product Principle leads to the generating function:

$$P(x) \;=\; \sum_{k\geq 0} p(k)x^k \;=\; \prod_{j\geq 1}\frac{1}{1-x^j}.$$

Let $\mathcal{Q}$ be the partitions into distinct parts: $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$, with $q(k) = \#\mathcal{Q}_k$.

A signed graded class $\mathcal{A}^\pm$ is endowed with a function $\mathrm{sgn} : \mathcal{A} \to \{\pm 1\}$. Its signed generating function is $A^\pm(x) = \sum_{k\geq 0}(\#\mathcal{A}_k^+ - \#\mathcal{A}_k^-)x^k$, where $\mathcal{A}_k^+ = \{a \text{ with } |a| = k \text{ and } \mathrm{sgn}(a) = +1\}$.

INVOLUTION PRINICIPLE: Let $I : \mathcal{A}^\pm \to \mathcal{A}^\pm$ be a size-preserving, sign-reversing involution: $I^{-1} = I$, $|I(a)| = |a|$, and $\mathrm{sgn}(I(a)) = -\mathrm{sgn}(a)$ *except* for elements of $\mathcal{F} = \{a \text{ with } I(a) = a\}$. Then the signed generating functions are equal: $A^\pm(x) \;=\; F^\pm(x)$.

1. Let $q_{\leq 2}(k)$ be the number of partitions of k with no part repeated three or more times, i.e. $m_i \leq 2$: for example $q_{\leq 2}(4) = 4$ counts $4,\ 3{+}1,\ 2{+}2,\ 2{+}1{+}1$, but not $1{+}1{+}1{+}1 = 1^4$. Also, let $p_{\nmid 3}(k)$ be the number of partitions of k with no part divisible by 3: for example $p_{\nmid 3}(4) = 4$ counts $4,\ 2{+}2,\ 2{+}1{+}1,\ 1{+}1{+}1{+}1$, but not $3{+}1$.

Prove $q_{\leq 2}(k) = p_{\nmid 3}(k)$ via equality of generating functions $Q_{\leq 2}(x) = P_{\nmid 3}(x)$.

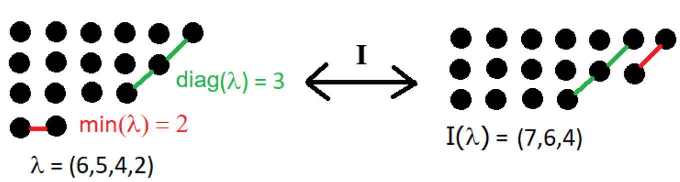
2. EULER PENTAGONAL NUMBER THEOREM: Let $\mathcal{Q}^\pm$ be the class of partitions into distinct parts with $\mathrm{sgn}(\lambda) = (-1)^{\ell(\lambda)}$. Its signed generating function is equal to:

$$Q^\pm(x) \;=\; \prod_{i\geq 1}(1-x^i) \;=\; 1 + \sum_{\ell\geq 1}(-1)^\ell(x^{\ell(3\ell+1)/2} + x^{\ell(3\ell-1)/2}).$$

We proved this using Franklin's involution I defined in terms of $\min(\lambda) = \lambda_\ell$, and:

$$\mathrm{diag}(\lambda) = \max(\{i \text{ with } \lambda_j = \lambda_1 - j + 1 \text{ for } j \leq i\}),$$

the length of the top right diagonal of the tableau λ. If $\min(\lambda) \leq \mathrm{diag}(\lambda)$, then I moves the bottom row of λ to a diagonal at the top right of $I(\lambda)$. If $\min(\lambda) > \mathrm{diag}(\lambda)$, then I reverses this, moving the upper-right consecutive diagonal to form a new bottom row.



If neither operation can be perfomed on λ within $\mathcal{Q}$, we let $I(\lambda) = \lambda$.

a. Verify that I is a size-preserving, sign-reversing involution staying within $\mathcal{Q}^\pm$.

b. Explicitly find the two sequences of fixed elements $\lambda = I(\lambda)$ with $|\lambda| = \frac{1}{2}\ell(3\ell\pm 1)$, and show there are no other fixed elements.

c. Use the equation $P(x)\cdot Q^\pm(x) = 1$ to recursively compute $p(k)$ for $k \leq 20$.

3. PRINCIPLE OF INCLUSION-EXCLUSION: For finite sets $\mathcal{B}_1, \ldots, \mathcal{B}_\ell \subset \mathcal{A}$, we have:

$$\#\Big(\mathcal{A} - \bigcup_{j=1}^{\ell} \mathcal{B}_j\Big) \;=\; \sum_{J\subset[\ell]} (-1)^{\#J} \, \# \bigcap_{j\in J} \mathcal{B}_j.$$

Re-prove this using the Involution Principle: $\#\widehat{\mathcal{A}}^+ - \#\widehat{\mathcal{A}}^- = \#\mathcal{F}^+ - \#\mathcal{F}^-$ on the signed set:

$$\widehat{\mathcal{A}}^\pm = \{(a, J) \text{ with } a \in \mathcal{A},\ J \subset [\ell],\ a \in \bigcap_{j\in J} \mathcal{B}_j\} \quad \text{and} \quad \operatorname{sgn}(a, J) = (-1)^{\#J}.$$

Find a sign-reversing involution I on $\widehat{\mathcal{A}}^\pm$ with fixed points:

$$\mathcal{F} = \{(a, \varnothing) \text{ with } a \in \mathcal{A} \setminus \bigcup_{j=1}^{\ell} \mathcal{B}_j\}.$$

HINT: Let $I(a, J) = (a, J')$ for some J' which is one larger or smaller than J.

4. A *derangement* is a permutation $w : [k] \xrightarrow{\sim} [k]$ with no fixed points: $w(i) \neq i$ for all i. Let D_k be number of derangements in S_k: e.g. $D_3 = 2$, counting $w = [231]$ and $[312]$, where we write $w = [w(1)\, w(2)\, w(3)]$.

a. Use PIE to find a formula for D_k, the number of derangements in S_k.

b. Use the formula to asymptotically estimate the fraction of $w \in S_k$ which are derangements.