

Math 880 Bigraded Classes, Stirling Cycle Numbers, Involutions Fall 2016

Bigraded classes. Following Flajolet-Sedgewick Ch. III, we define a *bigraded class* \mathcal{A} to be a set of combinatorial objects $a \in \mathcal{A}$ with *two* measures of magnitude, a primary measure $|a| = n$ called simply the “size”, and a secondary measure $\|a\| = k$ called the “weight” or “parameter”, or a “statistic on \mathcal{A} ”. Usually we consider *labeled* bigraded classes $\tilde{\mathcal{A}}$, in which each object $a \in \tilde{\mathcal{A}}$ has $n = |a|$ atoms having all the labels $1, 2, \dots, n$, as well as the weight $\|a\|$ unrelated to the labels: in particular, a permutation of the labels should not change either $|a|$ or $\|a\|$. We have the counting numbers $A_n^{(k)} = \#\{a \in \tilde{\mathcal{A}} \text{ with } |a| = n, \|a\| = k\}$.

The classic example is the labeled class $\tilde{\mathcal{P}} = \coprod_{n \geq 0} \mathfrak{S}_n$ comprising all permutations $w \in \mathfrak{S}_n$, with size $|w| = n$ and weight $\|w\| = \text{cyc}(w) =$ number of cycles of w . In this case the counting numbers are the *Stirling cycle numbers*: $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = P_n^{(k)} = \#\{w \in S_n \mid \text{cyc}(w) = k\}$.

We take the *bivariate generating function*:

$$\tilde{A}(x, t) = \sum_{n, k \geq 0} A_n^{(k)} \frac{x^n}{n!} t^k = \sum_{a \in \tilde{\mathcal{A}}} \frac{x^{|a|}}{|a|!} t^{\|a\|}.$$

All the constructions available for labeled graded classes extend to the bigraded case. In particular, we endow the labeled product $\tilde{\mathcal{A}} * \tilde{\mathcal{B}}$ with the additive weight function $\|(a_S, b_T)\| = \|a\| + \|b\|$, where a_S means $a \in \tilde{\mathcal{A}}$ with its atoms relabeled by the set S .

We have the Labeled Bigraded Product Principle: for $\tilde{\mathcal{C}} = \tilde{\mathcal{A}} * \tilde{\mathcal{B}}$, the bivariate generating function is $\tilde{C}(x, t) = \tilde{A}(x, t) \cdot \tilde{B}(x, t)$. The proof is very similar to the single-graded case:

$$\begin{aligned} \tilde{A}(x, t) \cdot \tilde{B}(x, t) &= \left(\sum_{p \geq 0} \sum_{i \geq 0} A_p^{(i)} \frac{x^p}{p!} t^i \right) \cdot \left(\sum_{q \geq 0} \sum_{j \geq 0} B_q^{(j)} \frac{x^q}{q!} t^j \right) \\ &= \sum_{n, k \geq 0} \left(\sum_{p=0}^n \sum_{i=0}^k \binom{n}{p} A_p^{(i)} B_{n-p}^{(k-i)} \right) \frac{x^n}{n!} t^k \\ &= \sum_{n, k \geq 0} C_n^{(k)} \frac{x^n}{n!} t^k = \tilde{C}(x, t). \end{aligned}$$

The second inequality uses the change of index variables $n = p + q$ and $k = i + j$, so that $\binom{n}{p} \frac{1}{n!} = \frac{1}{p!q!}$. The third equality is because each element $(a_S, b_T) \in \tilde{\mathcal{C}}_n^{(k)}$ corresponds to the choice of $S \subset [n]$, $a \in \tilde{\mathcal{A}}_p^{(i)}$, $b \in \tilde{\mathcal{B}}_{n-p}^{(k-i)}$.

Similarly, the constructions SEQ_j , SEQ , SET_j , SET , CYC_j , CYC can be performed on bigraded classes, and give the same formulas for the bivariate generating functions.

Counting cycles in permutations. The class of labeled cycles of length n is $\text{CYC}_n[\widetilde{1}]$, where $[\widetilde{1}]$ is the single-element labeled graded class, with bivariate exponential generating function $\frac{n!}{n} \frac{x^n}{n!} t = \frac{x^n}{n} t$, since a cycle is the same as a permutation of $[n]$ up to rotation equivalence. Here $\frac{x^n}{n!}$ indicates the size n , while $t = t^1$ indicates that a single cycle has weight 1.

Allowing cycles of any length gives generating function $\sum_{n \geq 1} \frac{x^n}{n} t = t \log\left(\frac{1}{1-x}\right)$. Realizing any permutation as a set of k labeled cycles gives the bivariate generating function of

$\tilde{\mathcal{P}} = \coprod_{n \geq 0} S_n = \text{SET}(\text{CYC}[\widetilde{1}])$:

$$\tilde{P}(x, t) = \sum_{n, k \geq 0} \binom{n}{k} \frac{x^n}{n!} t^k = \sum_{k \geq 0} \frac{1}{k!} \left(t \log\left(\frac{1}{1-x}\right) \right)^k = \exp\left(t \log\left(\frac{1}{1-x}\right)\right) = \frac{1}{(1-x)^t}.$$

We can get several interesting specializations of this bivariate function. First, taking $t = 1$ gives us the single-variable exponential generating function:

$$\tilde{P}(x) = \exp\left(\log\left(\frac{1}{1-x}\right)\right) = \frac{1}{1-x}.$$

Indeed, we can realize permutations either as sets of cycles or as labeled sequences: $\tilde{\mathcal{P}} \cong \text{SET}(\text{CYC}[\widetilde{1}]) \cong \text{SEQ}(\widetilde{1})$; and the above computation is the generating function version.

Next, fixing k and taking the t^k coefficient gives the single-variable generating function:

$$\tilde{P}^{(k)}(x) = \sum_{n \geq 0} \binom{n}{k} \frac{x^n}{n!} = \frac{1}{k!} \left(t \log\left(\frac{1}{1-x}\right) \right)^k.$$

This does not give an explicit formula for the Stirling cycle numbers, but it allows complex analytic methods to give the asymptotic approximation: $\binom{n}{k} \sim \frac{(n-1)!}{(k-1)!} (\log n)^{k-1}$ as $n \rightarrow \infty$. This means that for large n , the fraction of n -permutations which have k cycles is very close to $\frac{(\log n)^{k-1}}{n(k-1)!}$.

Finally, fixing n and taking the coefficient of $\frac{x^n}{n!}$ gives the generating function $P_n(t) = \sum_{k=1}^n \binom{n}{k} t^k$. The Taylor Coefficient Formula gives:

$$P_n(t) = \frac{d^n}{dx^n} \tilde{P}(x, t) \Big|_{x=0} = \frac{d^n}{dx^n} \left(\frac{1}{(1-x)^t} \right) \Big|_{x=0} = t(t+1)(t+2) \cdots (t+n-1) = t^{\bar{n}}.$$

That is:

$$\sum_{k=1}^n \binom{n}{k} t^k = t^{\bar{n}}.$$

Substituting $-t$ for t and factoring out signs changes rising powers to falling:

$$\sum_{k=1}^n (-1)^{n-k} \binom{n}{k} t^k = t(t-1) \cdots (t-n+1) = t^{\underline{n}}.$$

Compare this with our formula involving Stirling partition numbers:¹

$$\sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} t^k = t^{\bar{n}}.$$

These formulas mean that $(-1)^{n-k} \binom{n}{k}$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are the change-of-basis coefficients between the usual power basis $1, t, t^2, t^3, \dots$ for the polynomials in t , and the falling power basis $1, t, t^{\underline{2}}, t^{\underline{3}}, \dots$. Hence, if for any N we define lower-triangular matrices $M_1 = \left((-1)^{n-k} \binom{n}{k} \right)_{n,k=1}^N$

¹Bijjective proof. The right side $t^{\bar{n}}$ counts all functions $f : [n] \rightarrow [t]$ for $t \in \mathbb{N}$, each of which can be factored into a surjective function, $\text{surj}(n, k) = k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$; and a choice of image, $\binom{t}{k} = \frac{1}{k!} t^{\bar{k}}$.

and $M_2 = \left(\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}\right)_{n,k=1}^N$, then these are *inverse* matrices: $M_1 M_2 = I = (N \times N)$ identity matrix. This is equivalent to the formula:

$$\sum_{j \geq 0} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

Similarly, $M_2 M_1 = I$ is equivalent to:

$$\sum_{j \geq 0} (-1)^{j-k} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \begin{bmatrix} j \\ k \end{bmatrix} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

These formulas can also be proved by the Involution Principle, the first using the same $\infty/00$ Involution used for $t^n = \sum_{j=1}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} t^j$ on p. 5.

The second can be proved using the Lonely/Crowded Involution. The left side counts permuted set partitions (S, w) : an unordered partition $S = \{S_1, \dots, S_j\}$ with $S_1 \sqcup \dots \sqcup S_j = [n]$, and numbered lexicographically so that $\min(S_1) < \dots < \min(S_j)$, along with a permutation $w \in \mathfrak{S}_j$ with k cycles, and given the sign $\text{sgn}(S, w) = (-1)^{j-k}$. The involution will define a new permuted set partition (S', w') . Take the smallest $a \in [n]$ such that, for $a \in S_i$, the cycle of sets $S_i \cup S_{w(i)} \cup S_{w(w(i))} \cup \dots$ contains at least two elements. If $S_i = \{a\}$ is a singleton, then join it with the next set on its cycle: $S'_i = S_i \cup S_{w(i)}$ and $w'(i) = w(w(i))$. If S_i is not a singleton, split it into two sets along the same cycle: $S'_i = \{a\}$ and $S'_{w'(i)} = S_i - \{a\}$, with $w'(w'(i)) = w(i)$. This changes the sign $(-1)^{j-k}$ by incrementing/decrementing j while leaving n, k fixed. If there is no such a , then this is the unique fixed point consisting of all singleton sets and $w = \text{id}$. In the signed count $\sum_{j \geq 0} (-1)^{j-k} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \begin{bmatrix} j \\ k \end{bmatrix}$, I pairs up and cancels all terms except the fixed point, which gives the right side.

Cycle formula. We give another proof of:

$$t^n = \sum_{k=1}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

For a whole number value $t \in \mathbb{N}$, the left side t^n can be interpreted as the number of:

- injective functions $f : [n] \rightarrow [t]$
- proper t -colorings of the complete graph K_n
- sequences (a_1, \dots, a_n) of $a_i \in [t]$ with $a_i \neq a_j$ for $i < j$
- vectors in the hyperplane complement $\mathbb{F}_t^n - \bigcup_{i < j} H_{ij}$, where \mathbb{F}_t is a finite field, and H_{ij} is the hyperplane having the i, j coordinates equal.

We can obtain the right side from the fourth interpretation, by using Mobius inversion (generalized PIE) on the lattice $\mathcal{L}(\mathcal{B})$ of subspaces $V \in \mathbb{F}_t^n$ generated by the braid arrangement $\mathcal{B} = \bigcup_{i < j} H_{ij}$. We give this the partial order of *reverse* inclusion: $U \leq V$ means $U \supset V$, so that the minimal element is the entire space $\hat{0} = \mathbb{F}_t^n$. This poset is isomorphic

to Π_n , the set partitions $S = \{S_1, \dots, S_k\}$ with $S_1 \sqcup \dots \sqcup S_k = [n]$, ordered by refinement. Note that under this isomorphism, $\dim(V)$ is equal to $\ell(S) = k$.

Using the functions $f, g : \mathcal{L}(\mathcal{B}) \rightarrow \mathbb{Z}$ given by:

$$f(V) = \#V = t^{\dim(V)}, \quad g(V) = \#(V - \bigcup_{H \not\supset V} H)$$

where the union runs over all hyperplanes $H = H_{ij}$ not containing V . Then:

$$f(U) = \sum_{V \subset U} g(V) \iff g(U) = \sum_{V \subset U} \mu(U, V) f(V) = \sum_{V \subset U} \mu(U, V) t^{\dim(V)},$$

where $\mu(W, U)$ is the Mobius function of $\mathcal{L}(\mathcal{B})$ defined by $\mu(U, U) = 1$ and $\sum_{U \leq V \leq W} \mu(U, V) = 0$ for $U < W$. In particular:

$$t^n = \#(\mathbb{F}_t^n - \bigcup_{i < j} H_{ij}) = g(\hat{0}) = \sum_{V \in \mathcal{L}(\mathcal{B})} \mu(\hat{0}, V) t^{\dim(V)} = \sum_{S \in \Pi_n} \mu(\hat{0}, S) t^{\ell(S)}.$$

The above expression is called the *characteristic polyomial* of the subspace arrangement: in fact, the chromatic polynomial of any graph is equal to the characteristic polyomial of the corresponding graphical hyperplane arrangement.

Taking the t^1 term of the above expression, corresponding to the maximal element $\hat{1} = V = \mathbb{F}_t(1, \dots, 1)$, we find that $\mu(\hat{0}, \hat{1}) = \mu(\mathbb{F}_t^n, 0) = (-1)^{n-1} (n-1)!$, and from the product poset structure of the interval $[\hat{0}, S]$ for $S = \{S_1, \dots, S_k\}$ with $\ell(S) = k$ and $n_i = \#S_i$, we easily find that $\mu(\hat{0}, S) = (-1)^{n-k} \prod_i (n_i - 1)!$. Therefore:

$$t^n = \sum_{S \in \Pi_n} (-1)^{n-\ell(S)} \left(\prod_{i=1}^{\ell(S)} (n_i - 1)! \right) t^{\ell(S)}.$$

Now, we can construct any permutation by partitioning $[n]$ into subsets of size n_1, \dots, n_k , and putting the elements of each block into an n_i -cycle in one of $(n_i - 1)!$ ways. Thus, for a given $k = \ell(S)$, the above expression counts all permutations with of $[n]$ with k cycles, and we have:

$$t^n = \sum_{k=1}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

We obtain yet another formula for t^n from the second and third interpretations above. A conjunction of conditions $a_i = a_j$ can be regarded as a set of pairs $\{i, j\}$ in the powerset $\wp(\binom{[n]}{2})$, a Boolean poset. Ordinary PIE corresponding to the poset $\wp(\binom{[n]}{2})$ gives:

$$t^n = \sum_{G \subset K_n} (-1)^{e(G)} t^{c(G)},$$

where G runs over all graphs on n labeled vertices, with $e(G)$ edges and $c(G)$ connected components. Comparing this to the previous formula, each pair $\{i, j\}$ can be considered as a relation $i \sim j$, and a set of pairs generates an equivalence relation corresponding to a set partition in Π_n . Note that each $S \in \Pi_n$ corresponds to $c_{n_1} \dots c_{n_k}$ sets in $\wp(\binom{[n]}{2})$, where c_j is the number of connected graphs on j labeled vertices, so there are much fewer terms in

the Π_n formula, and much fewer still in the $k = 1, \dots, n$ formula.

Bijective proof of cycle formulas. We first prove the positive formula:

$$t^{\bar{n}} = \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

The left side counts permutation partitions: that is, an ordered set partition of $[n]$ into sets (S_1, \dots, S_k) , where S_i may be empty, along with (w_1, \dots, w_k) , where $w_i \in \mathfrak{S}_{S_i}$ is a permutation of the set S_i . (One may picture this as an arrangement of n distinct flags on t flagpoles.) The right side counts pairs $(w, f) \in \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$, where $\mathcal{F} = \{\text{functions } f : [n] \rightarrow [t]\}$ and \mathcal{F}^w means the functions invariant under w , i.e. $f(w(i)) = f(i)$ for all i , or f is constant on each cycle of w , so that $|\mathcal{F}^w| = t^{\text{cyc}(w)}$.

There is an easy bijection between these objects which proves the formula. Given a permutation partition $(S_1, \dots, S_k; w_1, \dots, w_k)$, let $f(j) = i$ for $j \in S_i$, and let $w = w_1 \cdots w_k$. Conversely, given (w, f) , let $S_i = f^{-1}(i)$, and let w_i be the permutation w restricted to S_i .

We may also consider this as an example of Polya's Method (or Burnside's Lemma) counting orbits of group actions. Dividing both sides by $n!$ gives:

$$\binom{\binom{t}{n}}{\binom{t}{n}} = \frac{1}{n!} \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

The left side counts multisets with n objects of t kinds, which is the quotient of \mathcal{F} under the natural action of \mathfrak{S}_n . Now Burnside's Lemma gives:

$$\left| \frac{\mathcal{F}}{\mathfrak{S}_n} \right| = \frac{1}{|\mathfrak{S}_n|} \sum_{w \in \mathfrak{S}_n} |\mathcal{F}^w|,$$

where \mathcal{F}^w is the set of functions invariant under w . This translates directly to the two sides of the multiset formula.

Finally, we use the 0/00 Involution to prove the signed formula:

$$t^{\bar{n}} = \sum_{k=1}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

Now the left side counts injective functions $\mathcal{E} = \{f : [n] \hookrightarrow [t]\}$, while the right side is the signed count of $\coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$, where we define $\text{sgn}(w, f) = (-1)^{n-k}$, where w has k cycles.

Now we define a sign-reversing Cycle Splicing Involution

$$I : \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w \rightarrow \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$$

with fixed-point set $(\coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w)^I = (\text{id}, \mathcal{E})$, which will prove the formula. For $(w, f) \in \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$ with f injective, let $I(w, f) = (w, f)$; since f is constant on all cycles of w , it must have only 1-cycles, so $w = e$ and $(w, f) \in (\text{id}, \mathcal{E})$. If f not injective, suppose $f(ai) = f(j)$ for minimal $i, j \in [n]$; then define $I(w, f) = (w \cdot (ij), f)$, multiplying w by the transposition (ij) , so that $w \cdot (ij) = (i'j') \cdot w$ for $i' = w(i)$, $j' = w(j)$. This reverses sign,

since if i, j lie in the same cycle of w , then $w \cdot (ij)$ cuts this cycle into two; if i, j lie on different cycles, then $w \cdot (ij)$ joins these cycles into one. Thus, I will cancel all non-injective $(w, f) \in \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$ on the right side of the formula, leaving only the injective (id, \mathcal{E}) on the left side. We have shown the involution changes the sign parity; this is also evident since the Bruhat length (minimal product of transpositions) for $w \in \mathfrak{S}_n$ is $\ell(w) = n - \text{cyc}(w) = n - k$. Note also that the right side is equal to:

$$\sum_{k=1}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k = \sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} n^{n-\ell(w)}$$

This argument may also be seen as an example of counting orbits, this time with signs. Dividing by $n!$ gives:

$$\binom{t}{n} = \frac{1}{n!} \sum_{k=1}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

The left side counts n -element subsets of $[t]$, which is the quotient of \mathcal{E} under the free action of \mathfrak{S}_n . Also $\mathcal{E} \cong (\text{id}, \mathcal{E}) = (\coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w)^I$, so by the Involution Principle:

$$\left| \frac{\mathcal{E}}{\mathfrak{S}_n} \right| = \frac{1}{|\mathfrak{S}_n|} |(\coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w)^I| = \frac{1}{|\mathfrak{S}_n|} \sum_{w \in \mathfrak{S}_n} (-1)^{n-\text{cyc}(w)} |\mathcal{F}^w|,$$

which clearly translates to the signed cycle formula.

Derangement involution. The derangement numbers D_k count the class \tilde{D}_k of derangement permutations $\pi \in \mathfrak{S}_k$ with $\pi(i) \neq i$ for all i , meaning the cycle decomposition of π has no 1-cycles. This can be constructed as $\tilde{D} = \text{SET}^\sim(\text{CYC}_{\geq 2}[1]x)$, so $\tilde{D}(x) = \exp(-\log(1-x) - x) = \frac{e^{-x}}{1-x}$, and the derangement number is $\tilde{D}_k = k! \sum_{i=0}^k \frac{(-1)^i}{i!} \approx k! e^{-1}$.

We can also show $\tilde{D}(x) = e^{-x}(1-x)^{-1}$ by a Max-Passing Involution. Let:

$$\tilde{\mathcal{C}}^\pm = \tilde{\mathfrak{N}}^\pm * \tilde{\mathcal{P}}, \quad \tilde{\mathcal{C}}^\pm(x) = e^{-x}(1-x)^{-1},$$

where $\tilde{\mathfrak{N}} = \{\emptyset, [1], [2], \dots\}$ with $\text{sgn}([n]) = (-1)^n$, and $\tilde{\mathcal{P}} = \{\pi \in \mathfrak{S}_k \text{ for } k \geq 0\}$ is permutations. The elements of $\tilde{\mathcal{C}}^\pm$ are (J, π_K) , where $J \sqcup K = [k]$ and π_K is a permutation of K . Define the involution $I(J, \pi_K) = (J', \pi_{K'})$ as follows. Let $L \subset K$ be the fixed points of π_K and $m = \max(J \cup L)$. Now if $m \in J$, let $J' = J \setminus \{m\}$, $K' = K \cup \{m\}$, inserting the 1-cycle (m) into π_K ; or if $m \in K$, let $J' = J \cup \{m\}$, $K' = K \setminus \{m\}$, removing the 1-cycle (m) from π_K ; or if m does not exist because $(J, \pi_K) = (\emptyset, \pi_{[k]})$ for some derangement $\pi_{[k]}$, let $I(J, \pi_K) = (J, \pi_K)$. Then the Involution Principle gives:

$$\tilde{D} \cong (\tilde{\mathcal{C}}^\pm)^I, \quad \tilde{D}(x) = \tilde{\mathcal{C}}^\pm(x) = e^{-x}(1-x)^{-1}.$$