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THE NUMBER OF TREES

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The mathematical theory of trees was first discussed by Cayley in 1857 (1). He was successful in finding recursion formulas for counting the number of trees or rooted trees having a finite number of vertices, where the number of branches at a vertex was not limited. Cayley also recognized the possibility of studying the chemical problem of isomers by making use of the notion of a tree, although a restriction on the number of branches that may occur at a vertex is necessary for the solution of this problem. In 1931 Henze and Blair (2) developed recursion formulas for counting the number of trees or rooted trees having the same finite number of vertices, where the number of branches at a vertex was allowed to be at most four, except for a root vertex, which was allowed to have at most three branches. This was the first solution to a problem of isomers in chemistry. The number of such trees with n vertices is precisely the same as the number of structurally isomeric, aliphatic hydrocarbons, i.e. the compounds of the molecular formula C_nH_{2n+2}. The number of such rooted trees with n vertices is precisely the same as the number of structurally isomeric, mono-substituted, aliphatic hydrocarbons, i.e. the compounds of the molecular formula $C_nH_{2n+1}X$, where X represents any chemical radical or atom different from hydrogen.

In his classic publication in 1937 G. Polya (3) developed a powerful method for treating the symmetries of certain types of geometrical configurations under a given permutation group. Using as generating functions, power series whose coefficients represent the number of different possible configurations with respect to this permutation group, methods were developed which yield functional equations for these generating functions. These functional equations contain implicitly recursion formulas for determining the coefficients and his analysis of the functional equations resulted in asymptotic expressions for the coefficients. In particular, Polya studied many problems of interest to chemists, obtaining the recursion formulas of Henze and Blair, and Cayley; but he also solved a wealth of other problems connected with chemical isomers. Although, in his publication Polya restricts himself to counting those trees and rooted trees which are of foremost interest to chemists, it is clear that his methods permit generalization to the counting of trees and rooted trees in the cases we have covered. But it is not apparent that his methods of analysis of the generating functions can be generalized to yield asymptotic values.

It seems, however, that although the machinery he has set up is powerful for the solution of some very general problems in symmetries of geometrical configurations, much of it is superfluous for the treatment of trees or of rooted trees

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alone. Accordingly, in this paper purely combinatorial methods are employed for the developing of relations between the generating functions. These methods enable one to study some general problems concerning the number of trees and of rooted trees and to find recursion formulas for counting these objects. Furthermore, the method used here for the counting of trees is new and interesting and considerably simpler than the methods used in the past. Also, general methods of analysis have been found which yield asymptotic values for the coefficients involved in each of the generating functions studied.

By a *tree* we shall mean any finite, connected, one-dimensional complex, without cycles. By a *vertex* we shall mean an end point of a line segment occurring in the tree. (Thus a point is considered as vertex even if only one or two line segments contain it as end point.) By a *rooted tree* we shall mean any tree in which exactly one vertex, called the *root*, is distinguished from all the other vertices in the tree. By the *ramification number* of a vertex we shall mean the number of line segments which have that vertex in common.

We shall call two trees, T and T', *homeomorphic* if and only if there is a one-to-one, bi-continuous transformation of T onto T', which maps the vertices of T onto vertices of T' and conversely. We shall call two rooted trees homeomorphic if and only if they are homeomorphic as trees in such a way that the root of one tree is mapped onto the root of the other and conversely.

Given a rooted tree T, then by removing an open line segment at the root of T, we split T into two parts. The part B which does not contain the root is called a *branch* of T. The removed line segment belongs to two vertices; one is the root of T, the other is a vertex of B. We consider B as rooted tree by designating this other vertex as root of B.

Several branches of T may be homeomorphic. The number of branches that are homeomorphic to one another is called the *multiplicity* of this type of branch. The following lemma is then obvious:

LEMMA. *Two rooted trees, T and T', are homeomorphic if and only if their branches are homeomorphic and occur with the same multiplicity.*

We shall consider only those trees and rooted trees where the ramification number of each vertex, except a root, is not greater than a certain arbitrarily selected, positive integer m. ($m = \infty$ is permitted and means there is no restriction imposed on the ramification numbers of these vertices.) For rooted trees we select arbitrarily another positive integer r and require that the ramification of the root be not greater than r. (Similarly, $r = \infty$ is permitted and means no restriction is imposed on the ramification number of the root.) Throughout the discussion we keep m fixed.

By $A_n^{(r)}$ we mean the number of nonhomeomorphic rooted trees with n vertices, where the ramification number of the root is not greater than r. Since $A_n^{(m-1)}$ will play a central role in the theory we put $A_n^{(m-1)} = A_n$ as an abbreviation. Also for formal reasons we introduce the empty tree with no vertices and put $A_0 = 1$.

We now define the following formal power series:

$$\varphi(x) = A_0 + A_1x + A_2x^2 + \cdots$$

$$\psi_r(x) = A_1^{(r)} + A_2^{(r)}x + A_3^{(r)}x^2 + \cdots \qquad \text{for } r > 0$$

$$\psi_0(x) = 1$$

$$G(t, x) = \prod_{\nu=0}^{\infty} (1 - tx^{\nu})^{-A_\nu}.$$

We develop $G(t, x)$ in powers of t:

$$G(t, x) = \prod_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} (-1)^{\mu} \binom{-A_\nu}{\mu} t^{\mu} x^{\nu\mu} = \prod_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \binom{A_\nu + \mu - 1}{\mu} t^{\mu} x^{\nu\mu}$$

which gives

$$G(t, x) = \sum_{r=0}^{\infty} g_r(x) t^r$$

where

$$g_0(x) = 1$$

and

$$g_r(x) = \sum_{n=0}^{\infty} \left(\sum_{\substack{\mu_0+\mu_1+\cdots=r \\ \mu_1+2\mu_2+\cdots=n}} \binom{A_0 + \mu_0 - 1}{\mu_0} \binom{A_1 + \mu_1 - 1}{\mu_1} \cdots \right) x^n,$$

for $r > 0$. We contend that $g_r(x) = \psi_r(x)$. In order to find $A_{n+1}^{(r)}$ we first select a point as root and attach to it r branches. We must select the branches from rooted trees where the ramification number of the root is $\leqq m - 1$, in order to satisfy the limiting condition imposed by m. Furthermore, we must have n for the total number of vertices occurring in the r branches so that if we select

μ_0 trees with no vertex

μ_1 trees with one vertex

μ_2 trees with two vertices

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. . . . .

such that $\sum_{i=0} \mu_i = r$ and $\sum_{i=1} i\mu_i = n$ then we get a rooted tree which is of the correct category to contribute to $A_{n+1}^{(r)}$. The number of selections of μ_i objects with possible repetitions from a total of A_i objects is

$$\binom{A_i + \mu_i - 1}{\mu_i}$$

and the product of these binomial coefficients over the index i gives, according to our lemma, the total number of rooted trees for our choice of $\mu_0, \mu_1, \cdots$. If we now sum these products over all μ_i satisfying $\sum_{i=0} \mu_i = r$ and $\sum_{i=1} i\mu_i = n$ we get, again referring to the lemma, the total number of rooted trees with $n + 1$ vertices where the ramification number of each of the vertices in the branches is bounded by m and the ramification number of the root is bounded by r. Since we have $g_0(x) = \psi_0(x) = 1$ and since we have just shown $g_r(x) = \psi_r(x)$ for $r > 1$ this gives

$$G(t, x) = \sum_{r=0}^{\infty} \psi_r(x)t^r . \tag{1}$$

On the other hand

$$G(t, x) = \exp\left(-\sum_{\nu=0}^{\infty} A_\nu \log (1 - tx^\nu)\right) = \exp\left(\sum_{\nu=0}^{\infty}\sum_{\mu=1}^{\infty} A_\nu x^{\nu\mu} \frac{t^\mu}{\mu}\right)$$

$$G(t, x) = \exp\left(\sum_{\mu=1}^{\infty} \frac{t^\mu}{\mu} \varphi(x^\mu)\right). \tag{2}$$

Hence, $$G(t, x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{\mu=1}^{\infty} \frac{t^\mu}{\mu} \varphi(x^\mu)\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{\Sigma\mu_i=n} \frac{1}{\mu_1 !\, \mu_2 ! \cdots} \left(\frac{\varphi(x)}{1}\right)^{\mu_1} \left(\frac{\varphi(x^2)}{2}\right)^{\mu_2} \cdots\right) t^{\mu_1+2\mu_2+\cdots}$$

so that

$$\psi_r(x) = \sum_{\Sigma i\mu_i=r} \frac{1}{\mu_1 !\, \mu_2 ! \cdots} \left(\frac{\varphi(x)}{1}\right)^{\mu_1} \left(\frac{\varphi(x^2)}{2}\right)^{\mu_2} \cdots .$$

Hence, in particular

$$\psi_1(x) = \varphi(x)$$
$$\psi_2(x) = \tfrac{1}{2}\varphi(x^2) + \tfrac{1}{2}(\varphi(x))^2$$
$$\psi_3(x) = \tfrac{1}{3}\varphi(x^3) + \tfrac{1}{2}\varphi(x^2)\varphi(x) + \tfrac{1}{6}(\varphi(x))^3.$$

Since $A_n^{(m-1)} = A_n$ we have (whether m is finite or infinite)

$$\psi_{m-1}(x) = \frac{1}{x}\varphi(x) - \frac{1}{x}$$

and in case m is finite we get the following relation for $\varphi(x)$:

$$\frac{1}{x}\varphi(x) - \frac{1}{x} = \sum_{\Sigma i\mu_i=m-1} \frac{1}{\mu_1 !\, \mu_2 ! \cdots} \left(\frac{\varphi(x)}{1}\right)^{\mu_1} \left(\frac{\varphi(x^2)}{2}\right)^{\mu_2} \cdots . \tag{3}$$

In case $m = \infty$ we put $\psi(x) = \psi_\infty(x) = 1 / x$ or $\varphi(x) = 1 + x\psi(x)$. Substituting in (2) we obtain

$$G(t, x) = \frac{1}{1-t} e^{(tx/1)\psi(x)+(t^2x^2/2)\psi(x^2)+\cdots}$$

So on one hand we have

$$(1 - t)G(t, x) = \prod_{\nu=1}^{\infty} (1 - tx^\nu)^{-A_\nu} = \exp\left(\sum_{\nu=1}^{\infty} \frac{(tx)^\nu}{\nu} \psi(x^\nu)\right)$$

and on the other hand (1) shows

$$(1 - t)G(t, x) = 1 + \sum_{r=1}^{\infty} (\psi_r(x) - \psi_{r-1}(x))t^r$$

hence

$$\exp\left(\sum_{\nu=1}^{\infty} \frac{(tx)^\nu}{\nu} \psi(x^\nu)\right) = 1 + \sum_{r=1}^{\infty} (\psi_r(x) - \psi_{r-1}(x))t^r.$$

Both sides of the preceding expression are infinite sums of formal power series in x and t and late terms of these sums contain only high powers of t as well as x. Therefore, we may substitute $t = 1$ on both sides and obtain an identity. The right hand side then indicates a formal limit process which has for limit $\psi(x)$ because for any partial sum

$$1 + \sum_{r=1}^{k} (\psi_r(x) - \psi_{r-1}(x)) = \psi_k(x)$$

and it is easy to see that $\psi_k(x)$ and $\psi(x)$ must have identical coefficients up to and including the term with x^{k+1}. Hence we get

$$\psi(x) = \exp\left(\sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu} \psi(x^\nu)\right). \tag{4}$$

Trees

In order to count the number T_n of (unrooted) trees with n vertices we first establish the following lemma. Given a tree T, let T_0 be a subtree having no two of its vertices similar under any homeomorphism of T onto itself. Given a vertex P of T, with P not in T_0; but let P be adjacent to a vertex Q of T_0. Assume σ a homeomorphism which maps T onto itself and assume of σ that $\sigma(P) = P'$ is in T_0. Then putting $\sigma(Q) = Q'$ and calling l the segment $P - Q$ and l' the segment $P' - Q'$ we contend:

LEMMA. *Either $l' = l$ or $Q' = Q$.*

PROOF. Taking out l splits T into two parts T_P, T_Q and taking out l' splits T into two parts $T_{P'}$, $T_{Q'}$. Since σ is a homeomorphism we know $\sigma(T_P) = T_{P'}$ and $\sigma(T_Q) = T_{Q'}$. We assume now $l' \neq l$ and shall prove $Q' = Q$. To that effect it is sufficient to prove that l' is in T_0, for then Q' would be in T_0 and since no two different vertices of T_0 are similar, we would know $Q' = Q$. We assume $l' \notin T_0$ and deduce a contradiction.

Now $l' \notin T_0$ and $P' \in T_0$ means $T_0 \subset T_{P'}$, hence $Q \in T_{P'}$. Since $l' \neq l$ and P is adjacent to Q we know $l \in T_{P'}$ hence $l \notin T_{Q'}$.

Simliarly, $l \notin T_0$ and $Q \in T_0$ means $T_0 \subset T_Q$, hence $P' \in T_Q$. Since $l' \neq l$ and P' adjacent to Q' we know $l' \in T_Q$, and in particular $Q' \in T_Q$.

But since $Q' \in T_Q$ and $l \notin T_{Q'}$, we know the subtree $T_{Q'}$ is contained within T_Q, $T_{Q'} \subset T_Q$. Furthermore, the fact that $l' \in T_Q$ and $P' \neq Q'$ means that $T_{Q'}$ is a proper subtree of T_Q; but this together with the finiteness of T contradicts the fact that σ is a homeomorphism such that $\sigma(T_Q) = T_{Q'}$.

If $l' = l$ then σ interchanges P and Q and the line l is a symmetry line. A tree can have at most one symmetry line.

Assume now T_0 is the largest subtree of T which contains no two similar vertices; then any vertex of T which is a neighbor to a vertex of T_0 is similar to a vertex of T_0, for otherwise T_0 would not be maximal of its kind. By induction on the number of intervening vertices we see that every vertex of T is similar to one of T_0. We contend every line segment of T (except a symmetry line) is similar to exactly one line segment of T_0. Since every vertex of T is similar to one of T_0 we may as well assume that one end, say Q, of the given line segment $P - Q$ lies on T_0. The other end P will then be adjacent to Q. If P is also on T_0 there is nothing to prove, for then $P - Q$ lies on T_0 and is of course similar to itself under the identity mapping. In case P is not on T_0 then we know P is similar to a vertex P' of T_0 (because T_0 is assumed to be maximal of its kind) and our lemma shows that the mapping which carries P into P' maps the segment $P - Q$ into the segment $P' - Q$ which is on T_0 (except in case $P - Q$ is a symmetry line).

Thus, if T_0 is a subtree of T and is maximal of our type it contains exactly one representative for each class of similar vertices of T and each class of similar lines of T. It is furthermore important that T_0 is itself a tree. Now the Euler characteristic of any tree, namely the number of vertices minus the number of segments is 1. Since this is true in T_0 we have the following theorem, which is a certain refinement of the Euler characteristic of a tree:

THEOREM. *In any tree the number of nonsimilar vertices minus the number of nonsimilar lines (symmetry line excepted) is the number one.*

Consequently, if we count the total number of nonsimilar vertices occurring among all trees with n vertices, subtract the total number of nonsimilar line segments (except symmetry lines) occurring among these trees, then each individual tree gives the contribution 1, so we get as result the total number of trees. The total number of nonsimilar vertices is just $A_n^{(m)}$. The total number of nonsimilar line segments (symmetry line excepted) is

$$\frac{1}{2} \sum_{\substack{i+j=n \\ i \neq j;\, i,j>0}} A_i^{(m-1)} A_j^{(m-1)} + \binom{A_{n/2}^{(m-1)}}{2} = \frac{1}{2} \sum_{\substack{i+j=n \\ i,j>0}} A_i A_j - \frac{1}{2} A_{n/2}$$

with $A_{n/2} = 0$ for n odd. Namely, the first member of the left hand side counts the number of trees with a stressed line such that removing the line gives two rooted trees with different numbers of vertices. In the sum each term is counted twice, hence the factor $\frac{1}{2}$. The second member counts the number of trees with a stressed line segment such that removing the line gives two rooted trees with the same number of vertices. This stressed line will never be a symmetry line, because no repetitions are allowed in the selection. In both members we are careful that the ramification number of every vertex in the united tree is bounded by m. Hence, if we define T_n as the number of trees with n vertices, we have

$$T_n = A_n^{(m)} - \tfrac{1}{2} \sum_{\substack{i+j=n \\ i,j>0}} A_i A_j + \tfrac{1}{2} A_{n/2} .$$

If we now define $\phi(x) = T_1 + T_2 x + T_3 x^2 + \cdots$ we get

(5) $$\phi(x) = \psi_m(x) - \tfrac{1}{2} x(\psi_{m-1}(x))^2 + \tfrac{1}{2} x\psi_{m-1}(x^2)$$

which holds for $m = \infty$, where we replace $\psi_\infty(x)$ by $\psi(x)$.

Analytic Behavior of the Power Series

We have

$$\psi(x) = \sum_{\nu=1}^{\infty} A_{\nu+1} x^\nu = \prod_{\nu=1}^{\infty} (1 - x^\nu)^{-A_\nu}$$

and the logarithmic derivative gives

$$\frac{\sum_{\nu=0}^{\infty} \nu A_{\nu+1} x^{\nu-1}}{\sum_{\nu=0}^{\infty} A_{\nu+1} x^\nu} = \frac{d}{dx}\left(\sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} A_\nu \frac{x^{\nu\mu}}{\mu}\right) = \sum_{\nu,\mu=1}^{\infty} \nu A_\nu x^{\nu\mu-1} .$$

Multiplying by $x\,\psi\,(x)$ we obtain

$$\sum_{n=1}^{\infty} nA_{n+1} x^n = \left(\sum_{\nu,\mu=1}^{\infty} \nu A_\nu x^{\nu\mu}\right)\left(\sum_{\rho=0}^{\infty} A_{\rho+1} x^\rho\right) = \sum_{n=1}^{\infty} \Big(\sum_{\substack{\nu\mu+\rho=n \\ \nu,\mu\geqq 1 \\ \rho\geqq 0}} \nu A_\nu A_{\rho+1}\Big) x^n$$

$$nA_{n+1} = 1A_1 \sum_{\substack{\mu\geq 1 \\ \mu\leq n}} A_{n+1-\mu} + 2A_2 \sum_{\substack{\mu\geq 1 \\ 2\mu\leq n}} A_{n+1-2\mu} + \cdots + nA_n A_1.$$

If we define $S_n^{(i)} = \sum_{\substack{\mu\geqq 1 \\ i\mu\leqq n}} A_{n+1-i\mu}$ for $i \leqq n$, $S_n^{(i)} = 0$ for $i > n$ then

$$S_{n-i}^{(i)} = \sum_{\substack{\mu\geqq 1 \\ i\mu\leqq n-i}} A_{n-i+1-i\mu} = \sum_{\substack{\mu\geqq 1 \\ i(\mu+1)\leqq n}} A_{n+1-i(\mu+1)} = S_n^{(i)} - A_{n+1-i}$$

or in other words

(6) $$S_n^{(i)} = S_{n-i}^{(i)} + A_{n+1-i}$$

which together with

(7) $$nA_{n+1} = 1A_1 S_n^{(i)} + 2A_2 S_n^{(2)} + \cdots + nA_n S_n^{(n)}$$

are valuable formulas for actual calculation of the A_n. Using (6) and (7) we are also able to show that $\psi(x)$ has a certain region around $x = 0$ in which it converges. The A_n are a monotonic increasing sequence of positive numbers so that if we replace all terms in $S_n^{(i)}$ by the first term A_{n+1-i} we get

$$S_n^{(i)} \leqq n/i\, A_{n+1-i}$$

since there are at most n/i terms in $S_n^{(i)}$. Hence, we get

$$A_{n+1} \leqq A_1A_n + A_2A_{n-1} + \cdots A_nA_1 .$$

If we now define the following sequence of numbers: $B_1, B_2, \cdots$ by putting $B_1 = 1$ and $B_{n+1} = \sum_{i+j=n+1} B_iB_j$ then a proof by induction yields immediately $A_n \leqq B_n$ for all n. If we put $f(x) = \sum_{\nu=1}^{\infty} B_\nu x^\nu$ then $f(x)$ solves the equation $y^2 - y + x = 0$ so

$$f(x) = \tfrac{1}{2}(1 - \sqrt{1 - 4x}) = \tfrac{1}{2} \sum_{\nu=1}^{\infty} \binom{\frac{1}{2}}{\nu} (-1)^{\nu-1} 4^\nu x^\nu$$

hence

$$A_n \leqq \frac{1}{2} \binom{\frac{1}{2}}{n} (-1)^{n-1} 4^n$$

so $\psi(x)$ converges absolutely in $|x| < \frac{1}{4}$ hence $\varphi(x)$ converges in the same region for m finite or infinite and so do the $\psi_r(x)$ for each r (whether m is finite or infinite), since the latter are only polynomials in the $\varphi(x)$. Let α be the radius of convergence of $\varphi(x)$ or $\psi(x)$ according to whether $m < \infty$ or $m = \infty$.

Since the functions $\varphi(x)$ $(m < \infty)$ and $\psi(x)$ $(m = \infty)$ are power series with positive coefficients we are certain that $x = \alpha$ will be a singularity. We succeed in showing that everywhere else on the circle of convergence, $|x| = \alpha$, the functions are analytic, hence permit continuation to functions analytic in a larger region. At $x = \alpha$ the functions have a ramification of order 1 hence are analytic functions of $\sqrt{x - \alpha}$, and the asymptotic estimates of the size of the coefficients are based upon these facts. To justify these assertions strong use is made of the fact that for real x, $0 < x < \alpha$, the functions have positive values and are monotonic increasing with x.

We now define

$$H_r(y_1, y_2, \cdots y_r) = \sum_{\sum_{i=1} i\mu_i = r} \frac{1}{\mu_1!\,\mu_2!\cdots} \left(\frac{y_1}{1}\right)^{\mu_1} \left(\frac{y_2}{1}\right)^{\mu_2} \cdots \quad \text{for} \quad r \geqq 0$$

and $H_r = 0$ for $r < 0$. Then the formalism by which (1) and (2) were obtained shows

$$e^{ty_1 + t^2(y_2/2) + t_3(y_3/3) + \cdots} = \sum_{\nu=0}^{\infty} t^\nu H_\nu(y_1, y_2, \cdots y_\nu).$$

By finding $\partial/\partial y_i$ of the relation above we obtain

$$\frac{\partial H_k}{\partial y_i} = \frac{1}{i} H_{k-i}. \tag{8}$$

In case $m < \infty$ the functional equation for $\varphi(x)$ is from (3) seen to be

$$\frac{1}{x}\varphi(x) - \frac{1}{x} = H_{m-1}(\varphi(x), \varphi(x^2), \cdots) = \psi_{m-1}(x).$$

If we differentiate with respect to x we get

$$\frac{1}{x}\varphi'(x) - \frac{1}{x^2}(\varphi(x) - 1) = \sum_{\nu=1}^{m-1} \frac{1}{\nu} H_{m-1-\nu}(\varphi(x), \varphi(x^2), \cdots)\varphi'(x^\nu)\cdot \nu x^{\nu-1}$$

which if we rearrange and express in terms of the $\psi_\nu(x)$ becomes

$$(9) \qquad \left(\frac{1}{x} - \psi_{m-2}(x)\right)\varphi'(x) = \frac{1}{x^2}(\varphi(x) - 1) + \sum_{\nu=2}^{m-1} x^{\nu-1}\varphi'(x^\nu)\psi_{m-1-\nu}(x).$$

Since for real x, $0 < x < \alpha$, $\varphi(x)$ and $\varphi'(x)$ have positive values > 1 we see that $1 / x - \psi_{m-2}(x) > 0$ in the same region. On the other hand since $\psi_{m-2}(x)$ is a dominant function for the geometric series we see

$$\frac{1}{x} - \psi_{m-2}(x) < \frac{1}{x} - \frac{1}{1-x} < 0 \quad \text{for real} \quad x > \tfrac{1}{2}.$$

Hence $\alpha \leqq \frac{1}{2}$ for all finite m, so certainly for $m = \infty$. The functional equations (3) and (4) for $\varphi(x)$ and $\psi(x)$ show that for $0 < x < \alpha$

$$\frac{1}{x}\varphi(x) \geqq \frac{1}{(m-1)!}(\varphi(x))^{m-1}, \qquad \psi(x) \geqq e^{x\psi(x)}$$

hence $\varphi(x) \leqq \sqrt[m-2]{\frac{(m-1)!}{x}}$, $\frac{\psi(x)}{\log \psi(x)} \leqq \frac{1}{x}$ so we see that $\varphi(x)$ (in case $m \geqq 3$) and $\psi(x)$ ($m = \infty$) are bounded on the real axis below α. Both $\varphi(x)$ and $\psi(x)$ are monotonic increasing with x so that

$$\lim_{x=\alpha} \varphi(x) = a \quad \text{and} \quad \lim_{x=\alpha} \psi(x) = a \quad \text{exist.}$$

The partial sums for $x = \alpha$ are monotonic and bounded by a, hence Abel's Theorem yields

$$\varphi(\alpha) = a \qquad 3 \leqq m < \infty$$

$$\psi(\alpha) = a \qquad m = \infty$$

and the values α, a satisfy the functional equations (3) and (4). Since $\alpha < 1$ we may assume that $\varphi(x^i)$ and $\psi(x^i)$ are known analytic functions of x for all $i \geqq 2$ and all $|x| \leqq \alpha$. If we now put $y = \varphi(x)$ or $y = \psi(x)$ as the case may be then we know y satisfies $\mathfrak{F}(x, y) = 0$ where

$$(10) \qquad \mathfrak{F}(x, y) = \begin{cases} H_{m-1}(y, \varphi(x^2), \varphi(x^3), \cdots) - \frac{1}{x}y + \frac{1}{x} & 3 \leqq m < \infty \\ e^{xy+(x^2/2)\psi(x^2)+(x^3/3)\psi(x^3)+\cdots} - y & m = \infty. \end{cases}$$

Furthermore, the functions $y = \varphi(x)$, $y = \psi(x)$ are unique analytic solutions for $|x| < \alpha$ and we know $\mathfrak{F}(\alpha, a) = 0$. Near $x = \alpha$, $y = a$ $\mathfrak{F}(x, y)$ is analytic in each variable separately. We may be sure that $\partial \mathfrak{F} / \partial y\ (\alpha, a) = 0$ for otherwise there would be a function of x which is solution of the equation $\mathfrak{F}(x, y) = 0$ and is analytic in a neighborhood of $x = \alpha$, $y = a$. This function would have to be $\varphi(x)$ or $\psi(x)$, which would contradict the fact that $x = \alpha$ is a singularity of these functions. If we refer to equations (8) and (10) we see that $\mathfrak{F}_y(\alpha, a) = 0$ gives

$$
\text{(11)} \qquad \begin{cases} \psi_{m-2}(\alpha) = \dfrac{1}{\alpha} \quad \text{in case} \quad 3 \leqq m < \infty \\[2ex] \psi(\alpha) = \dfrac{1}{\alpha} \quad \text{in case} \quad m = \infty. \end{cases}
$$

On the other hand referring to equations (8), (9) and (10) we see

$$
\mathfrak{F}_x(\alpha, a) = \begin{cases} \sum_{\nu=2}^{m=1} \alpha^{\nu-1} H_{m-1-\nu}(a, \varphi(\alpha^2), \cdots) + \dfrac{1}{\alpha^2}(a - 1) \neq 0 \\[2ex] (a + \alpha\psi(\alpha^2) + \alpha^2 \psi'(\alpha^2) + \cdots)\cdot\psi(\alpha) \neq 0 \end{cases}
$$

which shows that x is an analytic function of y in the neighborhood of $x = \alpha$, $y = a$ such that

$$
\left(\frac{dx}{dy}\right)_a = -\frac{\mathfrak{F}_y(\alpha, a)}{\mathfrak{F}_x(\alpha, a)} = 0.
$$

But

$$
\left(\frac{d^2 x}{dy^2}\right)_a = -\frac{\mathfrak{F}_x(\alpha, a)\cdot \mathfrak{F}_{y^2}(\alpha, a)}{(\mathfrak{F}_x(\alpha, a))^2} \neq 0
$$

because

$$
\mathfrak{F}_{y^2}(\alpha, a) = \begin{cases} H_{m-3}(a, \varphi(\alpha^2), \cdots) \neq 0 \\ \alpha^2 \psi(\alpha) = \alpha \neq 0 \end{cases}
$$

hence the inverse functions $y = \varphi(x)$ and $y = \psi(x)$ have a ramification of order 1 at $x = \alpha$.

We have shown that $\psi(x)$ and $\varphi(x)$, consequently $\psi_r(x)$ are absolutely convergent at $x = \alpha$. Since the power series have positive coefficients we know the series will be absolutely and uniformly convergent in $|x| \leqq \alpha$ consequently the series define continuous functions on $|x| = \alpha$. For any $x \neq \alpha$ but $|x| = \alpha$ the power series define values $y = \varphi(x)$, $y = \psi(x)$ and we know from the form of $\mathfrak{F}(x, y)$ that in the neighborhood of such a point (x, y) it is analytic in each variable separately. If we refer again to (10) we see that

$$
F_y(x, y) = \begin{cases} H_{m-2}(y, \varphi(x^2), \cdots) - (1/x) = \psi_{m-2}(x) - (1/x) \\ xe^{xy(x^2/2)\psi(x^2)+\cdots} - 1 = x\psi(x) - 1 \end{cases}
$$

by putting $y = \varphi(x)$ or $y = \psi(x)$. But because of cancellations for $|x| = \alpha$ $(x \neq \alpha)$ we have

$$\left|\psi_{m-2}(x) - \frac{1}{x}\right| \geqq \left|\frac{1}{x}\right| - |\psi_{m-2}(x)| > \frac{1}{\alpha} - \psi_{m-2}(\alpha) = 0$$

$$|x\psi(x) - 1| \geqq 1 - |x\psi(x)| > 1 - \alpha\psi(\alpha) = 0.$$

Hence, on the boundary of the circle $|x| = \alpha$ (except at α) we know

$$\mathfrak{F}_y(x, \varphi(x)) \neq 0$$

$$\mathfrak{F}_y(x, \psi(x)) \neq 0,$$

which shows that $\varphi(x)$ and $\psi(x)$ are analytic functions for these points on the circle of convergence of their defining power series. At $x = \alpha$ they are regular functions of $\sqrt{x - \alpha}$. Therefore, we may extend the region in which these functions are analytic to a circular region of radius larger than α, provided we make a cut in this region extending along the positive reals from $x = \alpha$. Inside this region we have

$$(12) \qquad \begin{cases} \varphi(x) = a + bt + R(x)t^2 \\ \psi(x) = \dfrac{1}{\alpha} + bt + R(x)t^2 \\ \phi(x) = A + Bt + Ct^2 + Dt^3 + R(x)t^4 \end{cases}$$

where $t = \sqrt{x - \alpha}$, and $A = \phi(\alpha)$, and we know in each case that $R(x)$ is a regular function of x inside this extended domain.

Determination of the Coefficients b, D

For $m < \infty$ we have around $x = \alpha$

$$\varphi(x) = a + bt + \cdots, t = \sqrt{x - \alpha}$$

and putting

$$\psi_r(x) = a_r + ct + \cdots, a_r = \psi_r(\alpha)$$

then

$$\frac{d}{dt}(\psi_r(x)) = c + \cdots = \frac{d}{dx}(\psi_r(x)) \cdot 2t.$$

If we compute only those terms of $d/dx\ (\psi_r(x))$ which are not bounded at α, we get

$$c + \cdots = (\psi_{r-1}(x)\varphi'(x) + \cdots)2t = \left(\psi_{r-1}(\alpha) \cdot \frac{b}{2t} + \cdots\right)2t$$

hence $c = b \cdot a_{r-1}$ so that

$$\psi_r(x) = a_r + ba_{r-1}t + \cdots.$$

From (9) we see that by putting $(1/x)\varphi(x) - (1/x) = \psi_{m-1}(x)$

$$\left(\frac{1}{x} - \psi_{m-2}(x)\right)\varphi'(x) = \frac{1}{x}\psi_{m-1}(x) + \sum_{\nu=2}^{m-1} x^{\nu-1}\varphi'(x^\nu)\psi_{m-1-\nu}(x)$$

which gives, if we develop around $x = \alpha$

$$\left(\frac{1}{\alpha} - \psi_{m-2}(\alpha) - b\psi_{m-3}(\alpha)t\right)\frac{b}{2t} + \cdots$$
$$= \frac{1}{\alpha}\psi_{m-1}(\alpha) + \sum_{\nu=2}^{m-1} \alpha^{\nu-1}\varphi'(\alpha^\nu)\psi_{m-1-\nu}(\alpha) + \cdots$$

where the terms omitted on each side all contain t to the first power at least. Remembering from (11) that $1/\alpha - \psi_{m-2}(\alpha) = 0$ we get by equating constant terms

$$(13) \qquad -\tfrac{1}{2}b^2 a_{m-3} = \frac{1}{\alpha}a_{m-1} + \sum_{\nu=2}^{m-1}\alpha^{\nu-1}\varphi'(\alpha^\nu)a_{m-1-\nu}$$

which gives b in case $m < \infty$.

In case $m = \infty$ we have around $x = \alpha$

$$\psi(\alpha) = \frac{1}{\alpha} + bt + \cdots$$

and $\psi'(x) = \psi(x)(x\psi'(x) + x^3\psi'(x^2) + \cdots + \psi(x) + x\psi(x^2) + \cdots)$ so that

$$(14) \qquad x\psi'(x)(1 - x\psi(x)) = \psi(x)\sum_{\nu=1}^{\infty}(x^\nu\psi(x^\nu) + x^{2\nu+2}\psi'(x^{\nu+1})).$$

Around $x = \alpha$ the last expression gives

$$\alpha\frac{b}{2t}(1 - \alpha\psi(\alpha) - \alpha bt) = \psi(\alpha)\sum_{\nu=1}^{\infty}(\alpha^\nu\psi(\alpha^\nu) + \alpha^{2\nu+2}\psi'(\alpha^{\nu+1})) + \cdots$$

where the omitted terms contain t to the first power at least. Remembering from (11) that $1 - \alpha\psi(\alpha) = 0$, we get by equating constant terms

$$(15) \qquad -\tfrac{1}{2}\alpha^2 b^2 = \frac{1}{\alpha}\left(1 + \sum_{\nu=2}^{\infty}(\alpha^\nu\psi(\alpha^\nu) + \alpha^{2\nu}\psi'(\alpha^\nu))\right)$$

which gives b in case $m = \infty$.

Whether m is finite or infinite, we have according to (5)

$$\phi(x) = \psi_m(x) - \tfrac{1}{2}x(\psi_{m-1}))^2 + \tfrac{1}{2}x\psi_{m-1}(x^2)$$

where $\psi_\infty(x) = \psi(x)$. Around $x = \alpha$

$$\phi(x) = A + Bt + Ct^2 + Dt^3 + \cdots$$

$$\phi'(x) = \frac{B}{2t} + C + \tfrac{3}{2}Dt + \cdots.$$

If $m < \infty$ we have the relation $\psi_{m-1}(x) = (1/x)(\varphi(x) - 1)$ so that

$$\frac{d}{dx}(x(\psi_{m-1}(x))^2) = \frac{d}{dx}\left(\frac{1}{x}(\varphi(x) - 1)^2\right) = 2\psi_{m-1}(x)\varphi'(x) - (\psi_{m-1}(x))^2.$$

Hence,

$$\phi'(x) = \sum_{\nu=1}^{m} x^{\nu-1}\varphi'(x^\nu)\psi_{m-\nu}(x) - \psi_{m-1}(x)\varphi'(x) + \tfrac{1}{2}(\psi_{m-1}(x))^2 + \cdots$$

where all the terms omitted are analytic functions of x in the neighborhood of $x = \alpha$. Consequently,

$$\phi'(x) = \tfrac{1}{2}(\psi_{m-1}(x))^2 + \sum_{\nu=2}^{m} x^{\nu-1}\varphi'(x^\nu)\psi_{m-\nu}(x) + \cdots$$

which shows $\phi'(x)$ is bounded as $x \to \alpha$, hence $B = 0$. We now compute $\phi''(x)$ omitting all terms which are bounded at $x = \alpha$

$$\begin{aligned}\phi''(x) &= \psi_{m-1}(x)\psi'_{m-1}(x) + \sum_{\nu=2}^{m} x^{\nu-1}\varphi'(x^\nu)\psi'_{m-\nu}(x) + \cdots \\ &= (\psi_{m-1}(x)\psi_{m-2}(x) + \sum_{\nu=2}^{m-1} x^{\nu-1}\varphi'(x^\nu)\psi_{m-1-\nu}(x))\varphi'(x) + \cdots \\ &= \left(\frac{1}{\alpha}a_{m-1} + \sum_{\nu=2}^{m-1}\alpha^{\nu-1}\varphi'(\alpha^\nu)a_{m-1-\nu}\right)\frac{b}{2t} + \cdots .\end{aligned}$$

If we refer to (13) we see $\phi''(x) = -\frac{1}{2}b^2 a_{m-3} \cdot b/2t + \cdots$ but on the other hand $\phi''(x) = \frac{3}{4}D/t + \cdots$ so that

$$D = -\tfrac{1}{3}\, b^3 a_{m-3} \neq 0. \tag{16}$$

In case $m = \infty$ we get according to (5) by omitting terms which are analytic functions of x at $x = \alpha$

$$\phi'(x) = \psi'(x)(1 - x\psi(x)) - \tfrac{1}{2}(\psi(x))^2 + \cdots$$

then using (14) we get

$$\begin{aligned}\phi'(x) &= \frac{1}{x}\psi(x)\sum_{\nu=1}^{\infty}(x^\nu\psi(x^\nu) + x^{2\nu+2}\psi'(x^{\nu+1})) - \tfrac{1}{2}(\psi(x))^2 + \cdots \\ &= \tfrac{1}{2}(\psi(x))^2 + \frac{1}{x}\psi(x)\sum_{\nu=2}^{\infty}(x^\nu\psi(x^\nu) + x^{2\nu}\psi'(x^\nu))\end{aligned}$$

which shows $\phi'(x)$ is bounded as $x \to \alpha$, hence $B = 0$. We now compute $\phi''(x)$, omitting all terms which are bounded at $x = \alpha$, giving

$$\phi''(x) = \psi'(x)\left(\psi(x) + \frac{1}{x}\sum_{\nu=2}^{\infty}(x^\nu\psi(x^\nu) + x^{2\nu}\psi'(x^{2\nu}))\right) + \cdots$$

so that

$$\frac{3}{4}\frac{D}{t} + \cdots = \frac{b}{2t}\left(\frac{1}{\alpha} + \frac{1}{\alpha}\sum_{\nu=2}^{\infty}(\alpha^\nu\psi(\alpha^\nu) + \alpha^{2\nu}\psi'(\alpha^\nu))\right) + \cdots .$$

If we refer to (15) we see

$$\frac{3}{4}\frac{D}{t} + \cdots = \frac{b}{2t}\left(-\tfrac{1}{2}\alpha^2 b^2\right) + \cdots$$

hence,

$$D = -\tfrac{1}{3}b^3\alpha^2 \neq 0. \tag{17}$$

Asymptotic Values of A_n, $A_n^{(r)}$ and T_n

Inside the circular region of radius larger than α, with a cut from $x = \alpha$ we have for m finite or infinite

$$\psi_r(x) = a_r + ba_{r-1}\, t + t^2 R(x)$$

and for m finite

$$\varphi(x) = a + bt + t^2 R(x)$$

$$\phi(x) = \phi(\alpha) + Ct^2 - \tfrac{1}{3}b^3 a_{m-3}\, t^3 + t^4 R(x)$$

and for m infinite

$$\psi(x) = a + bt + t^2 R(x)$$

$$\phi(x) = A + Ct^2 - \tfrac{1}{3}b^3\alpha^2 t^3 + t^4 R(x)$$

where $t = \sqrt{x - \alpha}$, $A = \phi(\alpha)$, $a_k = \psi_k(\alpha)$, b is determined by (13) or (15), and $a = \varphi(\alpha)$ or $a = \psi(\alpha)$ according to whether m is finite or infinite and $R(x)$ is in each case a regular function of x and is different for each function. If we now express the coefficients A_n, $A_n^{(r)}$, and T_n by means of Cauchy's integral formula, taking for integration path a circle of radius $r > \alpha$ lying inside the extended domain and running along the upper and lower lip of the cut, we obtain the following estimates:
for $3 \leqq m \leqq \infty$

$$A_n^{(r)} = \left| ba_{r-1}\binom{\frac{1}{2}}{n}\right| \alpha^{(3/2)-n} + O\left(\frac{\alpha^{-n}}{n^{5/2}}\right)$$

for $3 \leqq m < \infty$

$$A_n = \left| b\binom{\frac{1}{2}}{n}\right| \alpha^{(1/2)-n} + O\left(\frac{\alpha^{-n}}{n^{5/2}}\right)$$

$$T_n = \left| \tfrac{1}{3}b^3 a_{m-3}\binom{\frac{3}{2}}{n}\right| \alpha^{(5/2)-n} + O\left(\frac{\alpha^{-n}}{n^{7/2}}\right)$$

for $m = \infty$

$$A_n = \left| b\binom{\frac{1}{2}}{n}\right| \alpha^{(3/2)-n} + O\left(\frac{\alpha^{-n}}{n^{5/2}}\right)$$

$$T_n = \left| \tfrac{1}{3}b^3\binom{\frac{3}{2}}{n}\right| \alpha^{(9/2)-n} + O\left(\frac{\alpha^{-n}}{n^{7/2}}\right).$$

Since

$$\left|\binom{\frac{1}{2}}{n}\right| \sim \frac{1}{2\sqrt{\pi}\, n^{3/2}}, \qquad \left|\binom{\frac{3}{2}}{n}\right| \sim \frac{3}{4\sqrt{\pi}\, n^{5/2}}$$

we get by putting $|b| = \beta$

for $3 \leqq M \leqq \infty$
$$A_n^{(r)} \sim \frac{\beta a_{r-1}\, \alpha^{3/2}}{2\sqrt{\pi}} \cdot \frac{\alpha^{-n}}{n^{3/2}}$$

for $3 \leqq m < \infty$
$$A_n \sim \frac{\beta \alpha^{1/2}}{2\sqrt{\pi}} \frac{\alpha^{-n}}{n^{3/2}}$$

$$T_n \sim \frac{\beta^3 a_{m-3}\, \alpha^{5/2}}{4\sqrt{\pi}} \frac{\alpha^{-n}}{n^{5/2}}$$

for $m = \infty$

$$A_n \sim \frac{\beta \alpha^{3/2}}{2\sqrt{\pi}} \frac{\alpha^{-n}}{n^{3/2}}$$

$$T_n \sim \frac{\beta^3 \alpha^{9/2}}{4\sqrt{\pi}} \frac{\alpha^{-n}}{n^{5/2}}.$$

The Special Case $m = 3$

In case $m = 3$ the functional equation is of such a simple form that special methods yield the numbers α and b more easily. As a matter of fact since (3) gives

$$\frac{1}{x}\varphi(x) - \frac{1}{x} = \tfrac{1}{2}\varphi(x^2) + \tfrac{1}{2}(\varphi(x))^2 \tag{18}$$

we get by substituting in (18) $\mu(x) = 1 - x\varphi(x)$ or $\varphi(x) = (1 - \mu(x))/x$

$$\frac{1}{x}\frac{1-\mu(x)}{x} - \frac{1}{x} = \tfrac{1}{2}\frac{1-\mu(x^2)}{x^2} + \tfrac{1}{2}\frac{1 - 2\mu(x) + (\mu(x))^2}{x^2},$$

which simplifies to the equation

$$\mu(x^2) = (\mu(x))^2 + 2x. \tag{19}$$

This is a functional equation that has been studied in detail by Wedderburn (4).

Since we found that for any m $\psi_1(x) = \varphi(x)$ and for m finite $\psi_{m-2}(\alpha) = 1/\alpha$ we get, in case $m = 3$, that $\varphi(\alpha) = 1/a$ and hence $\mu(\alpha) = 0$. Consequently, $\mu(\alpha^2) = 2\alpha$, $\mu(\alpha^4) = 6\alpha^2$, etc. Assume we know the numbers $c_0 = 2, c_1, c_2, \cdots, c_{n-1}$ such that

$$\mu(\alpha^{2^k}) = c_{k-1}\,\alpha^{2^{k-1}} \quad \text{for} \quad k \leqq n$$

then

$$\mu(\alpha^{2^{n+1}}) = c_{n-1}^2\,\alpha^{2^n} + 2\alpha^{2^n} = (c_{n-1}^2 + 2)\alpha^{2^n}.$$

Hence $c_0 = 2$, $c_n = c_{n-1}^2 + 2$ so that

$$c_0 = 2,\ c_1 = 6,\ c_2 = 38,\ c_3 = 1446,\ c_4 = 2{,}090{,}918,\ \cdots .$$

Now $\sqrt[2^n]{\mu(\alpha^{2^{n+1}})} = \alpha \cdot \sqrt[2]{c^n}$ and since $\lim_{n=\infty} \mu(\alpha^{2^n}) = \mu(0) = 1$ we see that $1/\alpha = \lim_{n=\infty} \sqrt[2^n]{c_n}$. Since $\mu(\alpha^{2^{n+1}}) = 1 - \alpha^{2^{n+1}} - \cdots$

$$\sqrt[2^n]{\mu(\alpha^{2^{n+1}})} = 1 - \frac{\alpha^{2^{n+1}}}{2^n} - \cdots \cong 1 - \frac{1}{c_{n-1}\, 2^n}$$

so the error in putting $1/\alpha = \sqrt[2^n]{c_n}$ is approximately $1/(c_{n+1}2^n)$.

Around $x = \alpha$ we have $\varphi(x) = (1/\alpha) + bt + \cdots$ so $\mu(x) = 1 - x\varphi(x) = 1 - \alpha \cdot 1/\alpha - \alpha bt - \cdots$ hence $-\alpha b = \lim_{x=\alpha} \dfrac{\mu(x)}{\sqrt{x - \alpha}}$. Now

$$\frac{(\mu(x))^2}{x - \alpha} = \frac{\mu(x^2) - 2x}{x - \alpha} = -2 + \frac{\mu(x^2) - \mu(\alpha^2)}{x - \alpha}$$

because $\mu(\alpha^2) = 2\alpha$. Therefore $\lim_{x=\alpha} \dfrac{(\mu(x))^2}{x - \alpha} = -2 + 2\alpha\mu'(\alpha^2)$. Since, as we see from (19) $x\mu'(x^2) = \mu(x)\mu'(x) + 1$ we get

$$\alpha^{2^n} \mu'(\alpha^{2^{n+1}}) = 1 + c_{n-1}\, \alpha^{2^{n-1}} \mu'(\alpha^{2^n})$$
$$= 1 + c_{n-1} + (c_{n-1}c_{n-2}) + \cdots (c_{n-1}c_{n-2} \cdots c_0\alpha\mu'(\alpha^2))$$

hence

$$-\alpha\mu'(\alpha^2) = \frac{1}{c_0} + \frac{1}{c_0\, c_1} + \frac{1}{c_0\, c_1\, c_2} + \cdots$$

so that

$$\lim_{x=\alpha} \frac{(\mu(x))^2}{x - \alpha} = -2 - \frac{2}{c_0} - \frac{2}{c_0\, c_1} - \cdots = -3 - \frac{1}{c_1} - \frac{1}{c_1\, c_2} - \cdots .$$

Now b is pure imaginary and should have positive imaginary part because $\varphi(x)$ is monotonic increasing as $x \to \alpha$ along the real axis. Consequently,

$$-\alpha b = \lim_{x=\alpha} \frac{\mu(x)}{\sqrt{x - \alpha}} = -\sqrt{-3 - \frac{1}{c_1} - \frac{1}{c_1\, c_2} - \cdots} .$$

Using these methods and putting $1/\alpha = \sqrt[24]{2090918}$ we obtain easily

$$\alpha = 0.4026975$$

$$\beta = 6.1603212.$$

The following table has been prepared to show how the asymptotic values for A_n and T_n agree with those obtained by actual count using the recursion formulas for the special cases $m = 4$ and $m = \infty$. The asymptotic value for A_n is denoted by $\tilde{A}_n$, similarly for T_n. The data in the case $m = 4$ were taken from the tables compiled by Henze and Blair. The functional equation to be solved in the case $m = 4$ is given below

$$\frac{1}{x}\varphi(x) - \frac{1}{x} = \tfrac{1}{3}\varphi(x^3) + \tfrac{1}{2}\varphi(x^2)\varphi(x) + \tfrac{1}{6}(\varphi(x))^3$$

where the coefficient of x^n in the power series which defines $\varphi(x)$ in the neighborhood of the origin is the number of structurally isomeric, mono-substituted, aliphatic hydrocarbons with n carbon atoms. Hence, around $x = 0$ $\varphi(x)$ has the development

$$\varphi(x) = 1 + x + x^2 + 2x^3 + 4x^4 + 8x^5 + 17x^6 + 39x^7 + \cdots .$$

It is remarkable that even from the very beginning the estimates are quite reliable.

	$m = 4$; $\alpha = 0.3551817$; $a = 2.117421$; $\beta = 3.080388$				$m = \infty$; $\alpha = 0.3383219$; $\beta = 7.924780$			
	$\frac{\beta\sqrt{\alpha}}{2\sqrt{\pi}} = 0.5178760$		$\frac{\beta^3\alpha^{5/2}a}{4\sqrt{\pi}} = 0.6563190$		$\frac{\beta\alpha^{3/2}}{2\sqrt{\pi}} = 0.4399237$		$\frac{\beta^3\alpha^{9/2}}{4\sqrt{\pi}} = 0.5349485$	
	A_n	$\tilde{A}_n$	T_n	$\tilde{T}_n$	A_n	$\tilde{A}_n$	T_n	$\tilde{T}_n$
1	1	1	1	1	1	1	1	2
2	1	1	1	1	1	1	1	1
3	2	2	1	1	2	2	1	1
4	4	4	2	1	4	4	2	2
5	8	8	3	2	9	9	3	2
10	507	513	75	65	719	708	106	86
15	48,965	49,363	4,347	4,170	87,811	86,965	7,741	7,050
18	830,219	838,099	60,523	59,008	1,721,159	1,708,440	123,867	114,875

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