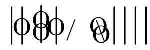
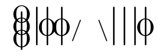
Math 481 **Quiz 10 - Solution** **Feb 5, 2025**

Six identical rings oooooo are arranged on 10 fingers ||||/ \||||, *at most two* on any finger.

Examples GOOD:  BAD: 

PROBLEM: Find the number of good arrangements, briefly explaining your method.
You may leave your answer in terms of symbols like $\binom{n}{k}$, $\left(\!\binom{n}{k}\!\right)$, etc

Standard Solution: Let

$$A = \{\text{all arrangements of 6 rings on 10 fingers}\}.$$

The Multiplicity Transform takes this to multisets of 10 objects from 4 kinds: the above examples become $\{2,3,3,4,6,6\}$ and $\{1,1,1,3,4,10\}$. Thus $|A| = \left(\!\binom{10}{6}\!\right)$.

Let the bad sets be $B_1, \ldots, B_{10}$ with:

$$B_i = \{\text{arrangements with} \geq 3 \text{ on finger } i\}.$$

To count $|B_i|$, we first put 3 rings on finger i, then arrange the remaining 3 rings on any of the 10 fingers, including extras on finger i, so $|B_i| = \left(\!\binom{10}{3}\!\right)$. Similarly $|B_i \cap B_j| = \left(\!\binom{10}{0}\!\right) = 1$, and there are no further intersections. By PIE, the number of good arrangements is:

$$\begin{aligned} \left|A - \textstyle\bigcup_{i=1}^{10} B_i\right| &= |A| - \sum_i |B_i| + \sum_{i<j} |B_i \cap B_j| - \cdots \\ &= \left(\!\binom{10}{6}\!\right) - \binom{10}{1}\left(\!\binom{10}{3}\!\right) + \binom{10}{2}\left(\!\binom{10}{0}\!\right) = 2850. \end{aligned}$$

Alternative Solution: Define bad sets $B'_1, \ldots, B'_4$ by:

$$B'_i = \{\text{arrangements with exactly } i+2 \text{ on any finger}\}.$$

To count B'_2, we pick a finger and put 4 rings on it, then arrange the remaining 2 rings on the other 9 fingers. Thus $|B'_2| = 10\left(\!\binom{9}{2}\!\right)$ and similarly $|B'_3| = 10\left(\!\binom{9}{1}\!\right)$, $|B'_4| = 10\left(\!\binom{9}{0}\!\right)$.

However, B'_1 is more complicated. We pick a finger to have 3 rings, then distribute the remaining 3, but we might put all 3 on the same other finger. This would lead to double-counting such a case, once for each 3-ring finger, so we need to subtract out the extra copy of such cases:

$$|B'_1| = 10\left(\!\binom{9}{3}\!\right) - \binom{10}{2}\left(\!\binom{8}{0}\!\right).$$

There are no intersections $B'_i \cap B'_j$, so:

$$\left|A - \textstyle\bigcup_{i=1}^{4} B'_i\right| = \left(\!\binom{10}{6}\!\right) - \left(10\left(\!\binom{9}{3}\!\right) - \binom{10}{2}\left(\!\binom{8}{0}\!\right)\right) - 10\left(\!\binom{9}{2}\!\right) - 10\left(\!\binom{9}{1}\!\right) - 10\left(\!\binom{9}{0}\!\right) = 2850.$$

Moral: When using PIE, the key is to choose the bad sets as similar as possible to each other, by a simple bad condition that is easy to count, as in the Standard Solution. This moves the work into mechanically computing a complicated algebraic formula, but without the combinatorial subtleties needed for the Alternative Solution.

Extra Credit: Equating the two solutions, we find that $\left(\!\binom{10}{3}\!\right) = \left(\!\binom{9}{3}\!\right) + \left(\!\binom{9}{2}\!\right) + \left(\!\binom{9}{1}\!\right) + \left(\!\binom{9}{0}\!\right)$. Can you generalize and explain this?