Math 411.002 Linear Mappings, Twofold Symmetry Fall 2020

Linear mappings and matrices. We consider a linear mapping

$$L : \mathbb{R}^2 \to \mathbb{R}^2,$$

which we can picture as a motion of the plane, moving each vector v to $L(v) = w$, for example stretching vectors to twice their length from the origin. *Linear* means that L respects vector addition and scalar multiplication:

$$L(v_1 + v_2) = L(v_1) + L(v_2) \qquad \text{and} \qquad L(rv) = rL(v) \quad \text{for} \quad r \in \mathbb{R}.$$

Thus, once we know the outputs of the basis vectors $L(1,0) = (a,b)$ and $L(0,1) = (c,d)$, we can compute $L(x,y)$ for any $v = (x,y)$:

$$\begin{aligned} L(x,y) &= L\big(x(1,0) + y(0,1)\big) = xL(1,0) + yL(0,1) \\ &= x(a,b) + y(c,d) = (ax+cy, bx+dy). \end{aligned}$$

The *matrix* of L, denoted $[L]$, records the outputs of the basis in its columns:

$$[L] = \left[\begin{array}{c|c} | & | \\ L(1,0) & L(0,1) \\ | & | \end{array}\right] = \left[\begin{array}{c|c} a & c \\ b & d \end{array}\right].$$

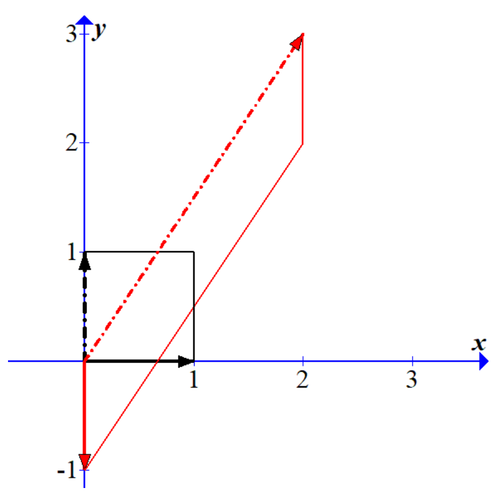
We define matrix multiplication of $[L]$ times the column vector $v = (x,y) = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ to give the output $L(v)$:

$$L(x,y) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+cy \\ bx+dy \end{bmatrix}.$$

Example: Let $L(x,y) = (2y, -x-3y)$, so that:

$$L(1,0) = (0,-1), \quad L(0,1) = (2,3), \quad [L] = \left[\begin{array}{c|c} 0 & 2 \\ -1 & 3 \end{array}\right].$$

We can picture this mapping as taking the unit square with sides $(1,0)$ and $(0,1)$ to the red image parallelogram with sides $(0,-1)$ and $(2,3)$:



Given two linear mappings L, M, we can do one after the other to get the composition mapping $L \circ M$ defined by: $(L \circ M)(v) = L(M(v))$. We define matrix multiplication so that it gives the matrix of the composition:

$$[L \circ M] \;=\; [L] \cdot [M].$$

This is the main meaning of matrix multiplication.

To avoid fussy notation we drop the brackets, so that L can denote both the linear mapping and its matrix, and $\cdot$ denotes composition of mappings as well as matrix multiplication.

Diagonalization. A better way to picture a linear mapping L is as stretching or flipping certain vectors. An *eigenvector* v is a nonzero vector which gets multiplied by a scalar λ, called the *eigenvalue* of v:

$$L(v) = \lambda v \quad \text{for} \quad \lambda \in \mathbb{R}.$$

If $\lambda > 0$, this means L stretches or shrinks v; if $\lambda < 0$, it also flips v over.

Given L, how to find such special vectors? First, we find the eigenvalues. Recall that a linear mapping M is *singular* when it crushes some non-zero vector to zero, and this is equivalent to the vanishing of the determinant:

$$\text{Some } v \neq 0 \text{ has } M(v) = 0 \quad \Longleftrightarrow \quad \det(M) \;=\; \det\begin{bmatrix} a & c \\ b & d \end{bmatrix} \;=\; ad - bc \;=\; 0.$$

(Geometrically, the determinant measures the area of the image parallelogram of M, which is zero when the paralleleogram collapses to a line segment.) Now, a scalar λ is an eigenvalue of some eigenvector when:

$$\begin{aligned} \text{Some } v \neq 0 \text{ has } L(v) = \lambda v \quad &\Longleftrightarrow \quad \text{some } v \neq 0 \text{ has } L(v) - \lambda I(v) \;=\; 0 \\ &\Longleftrightarrow \quad \text{some } v \neq 0 \text{ has } (L - \lambda I)(v) \;=\; 0 \\ &\Longleftrightarrow \quad \det(L - \lambda I) \;=\; 0, \end{aligned}$$

where I is the identity mapping $I(v) = v$. Thus, the eigenvalues are precisely the roots of the polynomial function $p(\lambda) = \det(L - \lambda I)$, the *characteristic polyonmial* of L, so we can find all the eigenvalues without knowning any eigenvectors! Once we have such a root λ, we can use Gaussian elimination to solve for the unknown vector v in the linear system $[L - \lambda I] \cdot v = 0$.

Once we have the eigenvalues λ_1, λ_2 and the basis of eigenvectors v_1, v_2, we can picture L as stretching these two axes by their respective eigenvalues, as if L were the diagonal matrix $D = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$. Indeed, if we form the change-of-basis matrix $P = [v_1 \,|\, v_2]$ with $P(1,0) = v_1$, $P(0,1) = v_2$, we get:

$$[L] \;=\; PDP^{-1} \;=\; P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P^{-1},$$

because:

$$PDP^{-1}(v_1) \;=\; P(D(P^{-1}(v_1))) \;=\; P(D(1,0)) \;=\; P(\lambda_1(1,0)) \;=\; \lambda_1 v_1 \;=\; L(v_1),$$

and similarly for $L(v_2)$.

EXAMPLE: Continuing with $L = \left[\begin{smallmatrix} 0 & 2 \\ -1 & 3 \end{smallmatrix}\right]$, the characteristic polynomial is:

$$p(\lambda) \;=\; \det(L{-}\lambda I) \;=\; \det\begin{bmatrix} -\lambda & 2 \\ -1 & 3{-}\lambda \end{bmatrix} \;=\; \lambda^2 - 3\lambda + 2.$$

This has roots $\lambda_1 = 1$, $\lambda_2 = 2$, so these must be the eigenvalues.

To find the eigenvector for $\lambda_1 = 1$, we must solve $(L{-}\lambda_1 I)(v_1) = 0$:

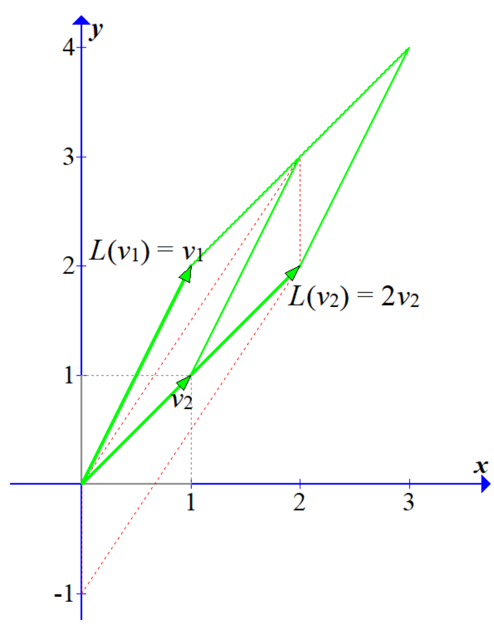
$$\begin{bmatrix} -\lambda_1 & 2 \\ -1 & 3{-}\lambda_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We row-reduce this system of two linear equations in two variables:

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \circlearrowleft -1 \quad\iff\quad \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} +1 \quad\iff\quad \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

That is, the system is equivalent to $x - 2y = 0$, or $v_1 = (x, y) = (2y, y)$. Taking $y = 1$ gives $v_1 = (2, 1)$, which indeed has $L(v_1) = \lambda_1 v_1 = v_1$. Similarly, we get the other eigenvector $v_2 = (1, 1)$ with $L(v_2) = \lambda_2 v_2 = 2v_2$.

$$L \;=\; \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \;=\; PDP^{-1} \;=\; \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$



Orthogonal mappings. We measure distances and angles in the plane $\mathbb{R}^2$ using the dot product, defined geometrically on vectors $v, w \in \mathbb{R}^2$ as $v \cdot w = |v||w| \cos(\theta_{vw})$, where $|\;|$ means length and θ_{vw} is the angle between vectors; and defined algebraically as:

$$v \cdot w \;=\; (a, b) \cdot (c, d) \;=\; ac + bd.$$

The dot product is different from the matrix product: if we consider v, w as $2{\times}1$ column vectors, we cannot multiply them. However, consider the *transpose* operation which exchanges rows with columns:

$$v^T \;=\; \begin{bmatrix} a \\ b \end{bmatrix}^T \;=\; [\; a \;\; b \;].$$

Then $v^T{\cdot}\, w$, the product of $1{\times}2$ and $2{\times}1$ matrices, is a $1{\times}1$ matrix, a scalar, and in fact it is the dot product: $v^T \cdot w = v \cdot w$.

Geometrically, two vectors are perpendicuar (orthogonal) when $v{\cdot}w = 0$, and v is a unit vector (length one) when $v \cdot v = 1$. The standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$ is *orthonormal*, meaning e_1, e_2 are are both unit vectors and are orthogonal to each other.

A linear mapping $R : \mathbb{R}^2 \to \mathbb{R}^2$ is *orthogonal* or *rigid* if it preserves distances and angles, taking any shape to a congruent image shape: for example a rotation. Clearly, an orthogonal mapping must take the standard basis to another orthonormal basis:

$$R(e_1) \cdot R(e_1) \;=\; R(e_2) \cdot R(e_2) \;=\; 1, \quad R(e_1) \cdot R(e_2) \;=\; R(e_2) \cdot R(e_1) \;=\; 0.$$

It turns out this is enough for R to be orthogonal, and this gives a simple matrix criterion for orthogonal mappings. Define transpose of a matrix to exchange all columns with rows:

$$R = \left[\begin{array}{c|c} a & c \\ b & d \end{array}\right], \quad R^T = \left[\begin{array}{cc} a & b \\ \hline c & d \end{array}\right].$$

Now, for $R(e_1) = (a, b)$, $R(e_2) = (c, d)$, the matrix product $R^T \cdot R$ is the matrix of all dot products of these columns:

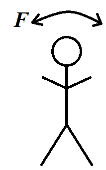
$$R = \left[\begin{array}{c|c} a & c \\ b & d \end{array}\right] \text{ is orthogonal} \quad \iff \quad R^T{\cdot}R \;=\; \left[\begin{array}{cc} a & b \\ \hline c & d \end{array}\right] \cdot \left[\begin{array}{c|c} a & c \\ b & d \end{array}\right] \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;=\; I.$$

That is, $R^T{\cdot}R = I$ and $R^{-1} = R^T$.

For example Rot_α, the counterclockwise rotation by angle α, has matrix:

$$\mathrm{Rot}_\alpha \;=\; \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad \mathrm{Rot}_\alpha^{-1} \;=\; \mathrm{Rot}_{-\alpha} \;=\; \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \;=\; \mathrm{Rot}_\alpha^T.$$

Twofold symmetry. A *symmetry* of an object X is an invertible mapping from X to itself which preserves the structure of X. If X is a geometric object in the plane, a symmetry is an orthogonal mapping $R : \mathbb{R}^2 \to \mathbb{R}^2$ with $R(X) = X$. The most familiar form of symmetry is *bilateral*, in which $R = F$ is a flip or reflection which exchanges the identical halves of X:



This has eigenvectors $F(e_1) = e_1$ and $F(e_2) = -e_2$ (if you tilt your head), and matrix $F = \left[\begin{array}{c|c} 1 & 0 \\ 0 & -1 \end{array}\right]$.

A general flip has $F(v_1) = v_1$ and $F(v_2) = -v_2$ for some orthonormal eigenvectors $v_1 = (c, s) = (\cos\alpha, \sin\alpha)$ and $v_2 = (-s, c)$. We compute its matrix using the eigenvector change-of-basis matrix $P = \left[\begin{array}{c|c} c & -s \\ s & c \end{array}\right]$:

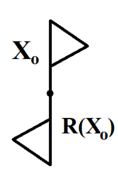
$$F \;=\; \mathrm{Fl}_\alpha \;=\; P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1} \;=\; \begin{bmatrix} c^2-s^2 & 2cs \\ 2cs & -c^2+s^2 \end{bmatrix} \;=\; \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

A flip F is a *twofold* symmetry, meaning that it is its own inverse, $F^{-1} = F$, so that $F(F(v)) = v$ or equivalently $F^2 = F{\cdot}F = I$. Is there any other kind of orthogonal twofold symmetry R with $R^2 = I$? For any such R, its eigenvalues $R(v) = \lambda v$ must satisfy $v = I(v) = R(R(v)) = \lambda^2 v$, so $\lambda^2 = 1$ and $\lambda = \pm 1$. Thus, the possible diagonal forms for R are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The first is the identity matrix, the trivial symmetry. The second is the bilateral symmetry above. The third is a $\frac{1}{2}$ rotation Rot_π. Note that in each case $R = R^{-1} = R^T$.

We construct an object X having $R = \mathrm{Rot}_\pi$ as its only non-trivial symmetry. Starting with a completely asymmetric object $X_\circ$, we combine it with its rotated image $R(X_\circ)$, producing a 2-bladed propeller or pinwheel $X = X_\circ \cup R(X_\circ)$.



We know R is a symmetry because:

$$R(X) \;=\; R(X_\circ) \cup R(R(X_\circ)) \;=\; R(X_\circ) \cup X_\circ \;=\; X.$$

Twofold symmetry in space. Any linear mapping $R : \mathbb{R}^3 \to \mathbb{R}^3$ has characteristic polynomial $p(\lambda) = \det(R - \lambda I)$ of degree 3, which must have a real root, and hence a real eigenvalue $\lambda \in \mathbb{R}$ with $R(v) = \lambda v$ for some eigenvector v. If R is orthogonal, we must have $|v| = |R(v)| = |\lambda||v|$, so that $|\lambda| = 1$ and $\lambda = \pm 1$.

If $\lambda = 1$ with no -1 eigenvalues, then R is a rotation around the axis v; if we complete $v_1 = v$ to an orthonormal basis $\{v_1, v_2, v_3\}$, and take the change-of-basis matrix $P = [\, v_1 \,|\, v_2 \,|\, v_3 \,]$, then v_1 is fixed and v_2, v_3 are rotated by some angle α:

$$R \;=\; P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} P^{-1}.$$

The other possibility is $\lambda = -1$, in which case:

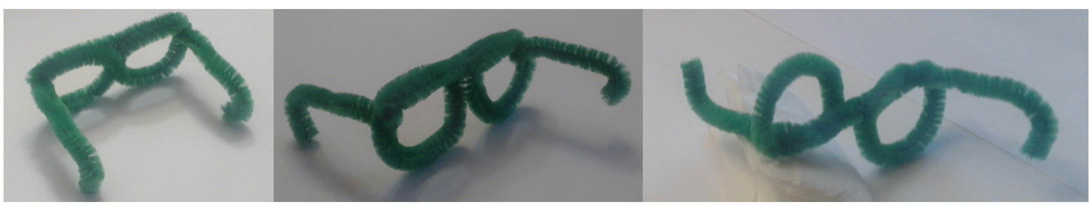
$$R \;=\; P \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} P^{-1}.$$

This is a *roto-reflection*: a rotation around an axis, combined with a reflection across the plane orthogonal to the axis; for $\alpha = 0$, it is just a reflection.

Now consider a twofold orthogonal symmetry, i.e. $R^2 = I$, $R^{-1} = R$. The non-trivial diagonal forms are:

$$R_1 \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_2 \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_3 \;=\; \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Each of these corresponds to a type of twofold symmetry in space, the symmetry of an object $X = X_\circ \cup R(X_\circ)$ for one of the $R = R_i$. The first is bilateral symmetry (reflection); the second is propeller symmetry (180° rotation); but the third is "point symmetry" (180° roto-reflection), a type seldom seen in natural or manufactured objects.



For examples in chemistry, the site `http://symmetry.otterbein.edu/gallery` has nice pictures of symmetric molecules; set Point Group Type to "C_1, C_s, C_i". The molecule cyclohexane-Br_2Cl_2 has point symmetry; click the picture to rotate around.