Collection of Methods and Theorems

Methods (Theorems at end)

Man vs. machine. We list detailed methods for standard problems from Calculus I & II. Some require connecting conceptual levels (physical, geometric, numerical, algebraic), and cannot be automated. However, it is important to know even those methods that are best done by computer (such as curve sketching and integration), so as to check at least the general shape of the answer for yourself. If you let the computer do the thinking, not just the calculating, you are ready to blindly accept any bizarre wrong answer, and one typo error can escalate into disaster. You must check the computer's answer against your own reasonable expectations.

§2.8. Method for related rates problems

1. Draw a picture labeled with:
   - numerical constant values
   - letter variables and their known current values (at time $t = 0$)
   - arrows showing known current rates of change (derivatives at $t = 0$)
   - an arrow for the unknown rate of change which is desired (the target rate)
2. Write an equation relating the variables according to the geometry of the picture.
3. Assuming each variable is a function of time t, take the derivative $\frac{d}{dt}$ of both sides of the equation, with the Chain Rule producing derivatives of the variables. If necessary, solve the derivative equation for the derivative which is desired.
4. Plug in the current values of the variables and rates to compute the target rate.

§3.7. Method for optimization. We aim to find the maximum or minimum possible value of a target quantity within the constraints of a (usually geometric) situation.

1. Draw a picture labeled with numerical constant values and with letters for varying quantities, including: *controlling* variables to determine the shape; *constrained* variables required to have a fixed value; and the *target* variable to be optimized.
2. Write equations relating variables according to the geometry of the picture.
3. Choose one of the controlling variables (say, x) as the *independent* variable, and write all other variables as functions of it by solving the above equations. Also determine the relevant domain $x \in [a, b]$, which is often restricted by requiring all variables to be positive.
4. Find the absolute maximum/minimum of the target variable over its domain, say $T = T(x)$ over $x \in [a, b]$. That is, solve $T'(x) = 0$ or undef, to find the critical points $x = c_1, c_2, \ldots$, as well as the endpoints $x = a, b$. Take the output values $T(x)$ at these candidate points: the largest/smallest output is the desired max/min.
5. If needed, find values of the other variables at the optimum x. Make sure the answer is physically plausible to check for mistakes.

§3.8 Newton's Method. For an equation $f(x) = 0$, find a numerical solution x with a specified accuracy, starting with a rough approximate solution $x \approx x_1$.

1. Numerically compute $x_2, x_3, \ldots$ according to the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, with at least the specified accuracy (number of decimal places).
2. Stop once the approximations no longer change: $x_n \approx x_{n+1}$ up to the given accuracy. The final approximate solution is $x \approx x_n$.

Notes by Peter Magyar `magyar@math.msu.edu`

§3.5. Method for Graphing. Given a function $y = f(x)$.

1. Determine the derivatives $f'(x)$ and $f''(x)$ with Derivative Rules.
   Determine the domain of $f(x)$: for what x the formula makes sense.
2. Solve $f'(x) = 0$ and $f'(x) = \text{undef}$ to find the critical points.
3. Make a sign table for $f'(x)$ to classify each critical point $x = a$:

		$x<a$	$x=a$	$x>a$
local max $\frown$	$f'(x)$	$+$	0	$-$
	$f(x)$	$\nearrow$	$f(a)$	$\searrow$
local min $\smile$	$f'(x)$	$-$	0	$+$
	$f(x)$	$\searrow$	$f(a)$	$\nearrow$
local max $\wedge$	$f'(x)$	$+$	undef	$-$
	$f(x)$	$\nearrow$	$f(a)$	$\searrow$
local min $\vee$	$f'(x)$	$-$	undef	$+$
	$f(x)$	$\searrow$	$f(a)$	$\nearrow$
vert asymp $\nearrow\vert\nwarrow$	$f'(x)$	$+$	$\frac{1}{0}$	$-$
	$f(x)$	$+$	$\frac{1}{0}$	$+$
vert asymp $\nearrow\vert\swarrow$	$f'(x)$	$+$	$\frac{1}{0}$	$+$
	$f(x)$	$+$	$\frac{1}{0}$	$-$
vert asymp $\searrow\vert\nwarrow$	$f'(x)$	$-$	$\frac{1}{0}$	$-$
	$f(x)$	$-$	$\frac{1}{0}$	$+$
vert asymp $\searrow\vert\swarrow$	$f'(x)$	$-$	$\frac{1}{0}$	$+$
	$f(x)$	$-$	$\frac{1}{0}$	$-$

   Here $f(a)$ means the output value is defined; and $\frac{1}{0}$ means a zero denominator at $x = a$ produces $\pm\infty$ values. There other possibilities if $x = a$ is a discontinuity (see §1.8).

4. Solve $f''(x) = 0$ or undef to find inflection points $x = a$; we also require that $f'(a)$ exists and is a local max/min of $f'(x)$. Make a sign table for $f''(x)$ if concavity is needed: $f''(x) > 0$ means concave up (smiling), $f''(x) < 0$ means concave down (frowning).
5. Solve $f(x) = 0$ to find the x-intercepts; and compute the y-intercept $(0, f(0))$.
6. Find the behavior as $x \to \pm\infty$.
   - Approximate by highest terms on top and bottom to get $f(x) \approx cx^p$.
   - For a better approximation of a rational function $f(x) = \frac{g(x)}{h(x)}$, use polynomial long division to get $f(x) = q(x) + \frac{r(x)}{h(x)}$.
     If $f(x) = mx + b + \frac{r(x)}{h(x)}$, then $y = mx + b$ is a slant asymptote.
     In general, $y = f(x)$ asymtotically approaches $y = q(x)$ as $x \to \pm\infty$.
7. Check for symmetries: ways to move the graph onto itself.
   - Side-to-side reflection symmetry for even function $f(-x) = f(x)$.
     EXAMPLES: $x^2{+}3$, x^4, $\cos(x)$
   - 180° rotation symmetry for odd function $f(-x) = -f(x)$.
     EXAMPLES: $2x$, x^3, $\sin(x)$
   - Shift-sideways translation symmetry for periodic $f(x{+}c) = f(x)$.
     EXAMPLES: $\cos(x{+}2\pi) = \cos(x)$, $\tan(x{+}\pi) = \tan(x)$.
8. Draw all the above features on the graph.

§3.9, 4.5, 6.6, 6.7, 7.1-7.4. Method for integration. For a function $f(x)$, find the indefinite integral $\int f(x)\,dx = F(x)+C$, i.e. an antiderivative function with $F'(x) = f(x)$. For brevity, we omit the constant $+C$.

1. Basic integrals which directly reverse basic derivatives:

$$\int x^p\,dx = \tfrac{1}{p+1}x^{p+1}\ (p\neq -1) \qquad \int \tfrac{1}{x}\,dx = \ln|x| \qquad \int e^x\,dx = e^x$$

$$\int \sin(x)\,dx = -\cos(x) \qquad \int \cos(x)\,dx = \sin(x)$$

$$\int \sec^2(x)\,dx = \tan(x) \qquad \int \tan(x)\sec(x)\,dx = \sec(x)$$

$$\int \tfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) \qquad \int \tfrac{1}{1+x^2}\,dx = \tan^{-1}(x) \qquad \int \tfrac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x)$$

$$\int \tfrac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}(x) = \ln(x+\sqrt{x^2-1}) \qquad \int \tfrac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}(x) = \ln(x+\sqrt{1+x^2})$$

2. Substitution: Factor the integrand so that $\int f(x)\,dx = \int h(g(x))\cdot g'(x)\,dx$.

   That is, find a factor $g'(x)$ which is a known derivative of some $g(x)$ appearing inside the other factor. To get $g'(x)$ exactly, perhaps multiply and divide by a constant. To find the outside $h(u)$, you may need to solve $u = g(x)$ as $x = g^{-1}(u)$.

   Take $u = g(x)$, $du = g'(x)\,dx$, so that $\int h(g(x))\cdot g'(x)\,dx = \int h(u)\,du = H(u)$. Restore the original variable: $\int f(x)\,dx = H(g(x))$.

3. Integration by Parts. Factor the integrand so that one factor is a known derivative $g'(x)$. Then: $\int f(x)\,dx = \int h(x)\cdot g'(x)\,dx = h(x)\cdot g(x) - \int g(x)\cdot h'(x)\,dx$.

   In Leibnitz notation, $\int u\,dv = uv - \int v\,du$.

   Do the remaining integral $\int g(x)\cdot h'(x)\,dx$ by another method. Here $g(x)$ should be *no more complicated* than $g'(x)$, and $h'(x)$ should be *simpler* than $h(x)$.

4. Products of Trig Functions. Substitute by factoring out a derivative $g'(x) = \cos(x), \sin(x), \sec^2(x)$ or $\tan(x)\sec(x)$; and writing the remaining factor in terms of $u = g(x)$ using $\cos^2(x)+\sin^2(x) = 1$, $\tan^2(x)+1 = \sec^2(x)$, $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

   Otherwise, use identities $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$, $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

   A hard case: $\int \sec(x)\,dx = \ln|\tan(x)+\sec(x)|$. Also, any trig integral converts into a rational function integral by the Tangent Half-Angle Substitution (§7.3).

5. Reverse Trig Substitution. If $\sqrt{a^2-x^2}$ appears in $\int f(x)\,dx$, complicate the integral by substituting $x = a\sin(\theta)$, $dx = a\cos(\theta)\,d\theta$; simplify using $\sqrt{a^2-(a\sin(\theta))^2} = a\cos(\theta)$. Do the resulting trig integral, then restore x using $\theta = \arcsin(\frac{x}{a})$.

   Do the same for $\sqrt{x^2-a^2}$ using $x = a\sec(\theta)$; and for $\sqrt{x^2+a^2}$ using $x = a\tan(\theta)$.

6. Partial Fractions for integrating rational functions $f(x) = \frac{g(x)}{h(x)}$, where $g(x), h(x)$ are polynomials. If $g(x)$ has degree greater than or equal to $h(x)$, perform long division to get $f(x) = q(x) + \frac{r(x)}{h(x)}$, where $r(x)$ has degree less than $h(x)$.

   If the denominator factors as $h(x) = (x-a)(x-b)\cdots$ with $a, b, \ldots$ all different, split $f(x)$ into the form: $f(x) = \frac{g(x)}{(x-a)(x-b)\cdots} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots$. Solve for the constant A after clearing denominators and substituting $x = a$; and similarly for the other constants $B, \ldots$. Finally, integrate using $\int \frac{A}{x-a}\,dx = A\ln|x-a|$.

   If $h(x)$ has factors like $(x-a)^k$ or ax^2+bx+c with no real roots, see §7.4.

§11.7. Method for Convergence Testing. For a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, determine if it converges toward a limit as we add more terms, or diverges (often to ∞).

0. If $\lim\limits_{n\to\infty} a_n \neq 0$, then the series diverges by the n-th Term Test (Vanishing Test).

1. Try to manipulate the series into a Standard Series:

   - Geometric series: $\sum\limits_{n=1}^{\infty} cr^{n-1} = c + cr + cr^2 + cr^3 + \cdots = \begin{cases} \frac{c}{1-r} & \text{for } |r| < 1 \\ \text{diverges} & \text{for } |r| \geq 1. \end{cases}$
   - Standard p-series: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots = \begin{cases} \text{converges} & \text{for } p > 1 \\ \text{diverges} & \text{for } p \leq 1. \end{cases}$

2. Estimate the fraction a_n by taking only the largest terms in the numerator and denominator, obtaining a simple b_n which is often a standard series. Convergence of $\sum a_n$ is likely to be the same as convergence of $\sum b_n$. Justify with a Test:

   - Direct Comparison Test (positive a_n)
     - Ceiling $0 \leq a_n \leq c_n$ where $\sum c_n$ converges $\implies$ $\sum a_n$ also converges.
     - Floor $0 \leq d_n \leq a_n$ where $\sum d_n$ diverges $\implies$ $\sum a_n$ also diverges.

     The ceiling c_n or floor d_n will usually be closely related to the estimate b_n.

   - Limit Comparison Test (positive a_n): Determine $L = \lim\limits_{n\to\infty} a_n / b_n$.
     - $L < \infty$ and $\sum b_n$ converges $\implies$ $\sum a_n$ also converges $[a_n < (L+\epsilon) b_n]$.*
     - $L > 0$ and $\sum b_n$ diverges $\implies$ $\sum a_n$ also diverges $[a_n > (L-\epsilon) b_n]$.

3. Try the Integral Test if a_n is positive and fairly simple, but not comparable to a standard series: e.g. $\frac{1}{n \ln(n)}$. For positive, decreasing, continuous $f(x)$ with $a_n = f(n)$, compute improper integral $\int_1^{\infty} f(x)\,dx = \lim\limits_{N\to\infty} \int_1^N f(x)\,dx = \lim\limits_{N\to\infty} F(N) - F(1)$.

   - $\int_1^{\infty} f(x)\,dx$ converges $\implies$ $\sum a_n$ also converges $[\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx]$.
   - $\int_1^{\infty} f(x)\,dx$ diverges $\implies$ $\sum a_n$ also diverges $[\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)\,dx]$.

4. Try the Ratio Test if a_n has a growing number of factors, for example if it contains r^n or $n!$. Determine $\lim\limits_{n\to\infty} |a_{n+1}/a_n| = L$.

   - $L < 1$ $\implies$ $\sum a_n$ converges $[a_n \leq c(L+\epsilon)^n]$.
   - $L > 1$ $\implies$ $\sum a_n$ diverges $[a_n \geq c(L-\epsilon)^n]$.
   - $L = 1$ $\implies$ no conclusion.

5. If $\sum a_n$ has positive and negative terms, try:

   - Absolute Convergence: $\sum |a_n|$ converges $\implies$ $\sum a_n$ also converges.
   - Alternating Series: For $a_n = (-1)^n b_n$ with $b_n \geq 0$

     $\lim\limits_{n\to\infty} b_n = 0$, b_n decreasing $\implies$ $\sum a_n$ converges.

     Error estimate: For $L = \sum_{n=1}^{\infty} a_n$, get $|L - \sum_{n=1}^{N} a_n| \leq b_{N+1}$ for $N \geq 1$.

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*Most later tests are proved by reducing to a Direct Comparison, specified in [brackets].

§5.1, 5.2, 5.4, 10.2, 10.4. Method of slice analysis to compute size. Let S be any measure of the size or bulk of a geometric object: length, area, volume, mass, etc. We want an integral formula to compute it.

1. Cut the object into slices whose position is determined by some variable $x \in [a, b]$.
2. Mark off the interval $[a, b]$ into n increments of width $\Delta x = \frac{b-a}{n}$, each with a sample point x_i. This splits the object into n slices, and summing up their sizes gives the total size: $S = \sum_{i=1}^{n} \Delta S_i$.
3. Because the slice at x_i is so thin, we can find a good approximation of its size by some simple formula of the form $\Delta S_i \approx f(x_i)\,\Delta x$.
4. Taking $n \to \infty$ and $\Delta x \to 0$, the approximations become exact:

$$S \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \;=\; \int_a^b f(x)\,dx\,.$$

5. Having expressed $S = \int_a^b f(x)\,dx$, we evaluate this integral by algebraic or numerical techniques.

Theorems. Key theoretical results from Calculus I & II.

§1.5 Limit definitions.

- $\lim_{x\to a} f(x) = L$ means that $f(x)$ can be forced arbitrarily close to L by making x sufficiently close to (but unequal to) a.
- $\lim_{x\to a} f(x) = \infty$ means that $f(x)$ can be forced to be arbitrarily large by making x sufficiently close to (but unequal to) a.
- $\lim_{x\to a^+} f(x) = L$ means that $f(x)$ can be forced arbitrarily close to L by making x sufficiently close to (but larger than) a.

§1.6 Squeeze Theorem: Suppose $f(x) \leq g(x) \leq h(x)$ for all x near a (except possibly $x = a$), and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$.

§1.8 Continuity definition. A function $f(x)$ is continuous at $x = a$ whenever $\lim_{x\to a} f(x) = f(a)$.
Graphically, a function is continuous whenever the graph $y = f(x)$ proceeds through the point $(a, f(a))$ without jumps or holes.

Types of discontinuity.

- Removable discontinuity: $f(a)$ is undefined, but $\lim_{x\to a} f(x)$ exists.
- Removable discontinuity: $f(a)$ and $\lim_{x\to a} f(x)$ exist, but are unequal.
- Jump discontinuity: the left and right limits are unequal, $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$.
- Vertical asymptote: $\lim_{x\to a^+} f(x)$ and/or $\lim_{x\to a^-} f(x)$ are $\pm\infty$.
- Essential discontinuity: $\lim_{x\to a^+} f(x)$ and/or $\lim_{x\to a^-} f(x)$ do not exist.

§1.8 Intermediate Value Theorem (IVT): If $f(x)$ is continuous for $x \in [a, b]$, and r is between $f(a)$ and $f(b)$, then there is a value $c \in (a, b)$ such that $f(c) = r$; that is, $f(x)$ must pass through every value r between $f(a)$ and $f(b)$.

§2.1 Derivative defintion: The *derivative* of $f(x)$ at $x = a$ means

$$f'(a) \;=\; \lim_{h\to 0} \frac{f(a{+}h) - f(a)}{h} \;=\; \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

The function is *differentiable* at $x = a$ if $f'(a)$ exists.

§2.2 Continuity Theorem. If $f(x)$ is differentiable at $x{=}a$, then $f(x)$ is also continuous at $x{=}a$.

§3.1 Extremal Value Theorem (EVT): If $f(x)$ is continuous on the closed, finite interval $x \in [a, b]$, then $f(x)$ possesses at least one absolute maximum point and one absolute minimum point.

§3.1 First Derivative Theorem: if $f(x)$ has a local maximum or minimum over $x \in [a, b]$ at $x = c \in (a, b)$, and $f'(c)$ exists, then $f'(c) = 0$.

§3.2 Mean Value Theorem (MVT): If $f(x)$ is continuous on the closed interval $x \in [a,b]$ and differentiable on the open interval $x \in (a,b)$, then there is some point $c \in (a,b)$ with

$$f'(c) \;=\; \frac{f(b)-f(a)}{b-a}\,.$$

That is, the tangent line to the graph $y = f(x)$ at some point $(c, f(c))$ must be parallel to the secant line from $(a, f(a))$ to $(b, f(b))$.

§3.2 Uniqueness Theorem: If $f(x), g(x)$ have the same derivative $f'(x) = g'(x)$ for all $x \in (a,b)$, and the same initial value $f(c) = g(c)$ for some $c \in [a,b]$, then $f(x) = g(x)$ for all $x \in [a,b]$. That is, there can be only one function with a given derivative and a given initial value.

§3.3 Increasing/Decreasing Theorem: Let $f(x)$ be continuous for $x \in [a,b]$.

- If $f'(x) > 0$ for all $x \in (a,b)$, then $f(x)$ is strictly increasing: $f(p) < f(q)$ for $p < q$.
- If $f'(x) \geq 0$ for all $x \in (a,b)$, then $f(x)$ is increasing: $f(p) \leq f(q)$ for $p < q$.
- Similarly for $f'(x) < 0$ and $f(x)$ decreasing.

§4.2 Integral Definition: Given a function $f(x)$ on an interval $x \in [a,b]$.

- Divide $[a,b]$ into n increments of width $\Delta x = \frac{b-a}{n}$, and choose sample points $x_1, \ldots, x_n$ with one in each increment: $x_i \in [a+(i-1)\Delta x, a+i\Delta x]$. Define the integral as a limit of *Riemann sums*:

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \;=\; \lim_{n\to\infty} f(x_1)\Delta x + \cdots + f(x_n)\Delta x.$$

- In an *upper Riemann sum*, choose the sample points so that $f(x_i)$ is maximal in its increment $x \in [a+(i-1)\Delta x, a+i\Delta x]$; then the sum gives an overestimate of the integral. Similarly for a *lower Riemann sum* giving an underestimate.
- The function $f(x)$ is *integrable* over $[a,b]$ whenever the above limit exists for every possible choice of sample points x_i.
- Theorem: Any continuous function, or even a function with a finite list of removable or jump discontinuities, is integrable over $[a,b]$.

§4.3 First Fundamental Theorem of Calculus (FTC1): Let $f(x)$ be continuous on $x \in [a,b]$ and define $I(x) \;=\; \int_a^x f(t)\,dt$. Then:

$$I'(x) \;=\; \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) \;=\; f(x)\,.$$

That is, the rate of change of the cumulative effect of $f(t)$ over $t \in [a,x]$ is the strength of the effect $f(x)$ at the endpoint $t = x$.

§4.3 Second Fundamental Theorem. If $f(x)$ has a known anti-derivative $F(x)$ with $F'(x) = f(x)$, then:

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a)\,.$$

That is, the cumulative effect of the rate of change $f(x) = F'(x)$ is the total change $F(b) - F(a)$.

§4.4 Average definition: The average of $f(x)$ over $x \in [a,b]$ is $f_{\text{ave}} \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx$.

§6.1 Inverse functions: Consider a function $f : A \to B$ with inputs in the set A and outputs covering the set B.

- Suppose f is one-to-one, meaning if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, that is, the graph $y = f(x)$ satisfies the horizontal line test.
- Define the *inverse function* $f^{-1} : B \to A$ as $f^{-1}(b) = a$, where $a \in A$ is the unique value with $f(a) = b$.
- The function f and its inverse f^{-1} undo each other: $f^{-1}(f(a)) = a \quad \text{and} \quad f(f^{-1}(b)) = b$ for all $a \in A$, $b \in B$.
- If $f(x)$ is differentiable at $x = a$, and $b = f(a)$, then $f^{-1}(y)$ is differentiable at $y = b$, and
$$(f^{-1})'(b) \; = \; \frac{1}{f'(a)} \; = \; \frac{1}{f'(f^{-1}(b))}.$$
In Leibniz notation with $y = f(x)$ and $x = f^{-1}(y)$: $\left.\frac{dx}{dy}\right|_{y=b} = \; 1 \,/ \left(\left.\frac{dy}{dx}\right|_{x=a} \right)$.

§6.8 L'Hôpital's Rule: For functions $f(x), g(x)$, suppose $f'(x), g'(x)$ exist and $g'(x) \neq 0$, on some interval $x \in (a-\delta, a+\delta)$. Suppose that either:

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty.$$

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \; = \; \lim_{x\to a} \frac{f'(x)}{g'(x)} \, ,$$

provided the right side limit exists, or equals ∞ or $-\infty$.

Let $f(x), g(x)$ be functions which are differentiable and $g'(x) \neq 0$, on a semi-infinite interval $x \in (c, \infty)$. Suppose that either:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0 \quad \text{or} \quad \lim_{x\to\infty} |f(x)| = \lim_{x\to\infty} |g(x)| = \infty.$$

Then:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \; = \; \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \, ,$$

provided the right side limit exists, or equals ∞ or $-\infty$. All this also holds with $x \to \infty$ replaced with $x \to -\infty$.