Peter Magyar

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Nutshell Version

1. Borel-Weil theorem for configuration varieties and Schur modules

One of the most useful constructions of algebra is the Schur module S_λ, an irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$, along with its character the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$. (Here $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ is a partition.) Combinatorists such as Lascoux-Schutzenberger, Stanley, and Kraskiewicz-Pragacz defined a menagerie of generalized Schur modules and polynomials with a variety of applications. These generalized modules S_D are associated to certain diagrams of squares in the plane, $D \subset \mathbb{Z} \times \mathbb{Z}$, with the original S_λ being the case when D is the Young diagram of λ.

Our paper investigates these modules from the Borel-Weil perspective, in which a family of linear representations is realized as the projective coordinate ring of a non-linear representation (i.e. a variety with $\mathrm{GL}_n(\mathbb{C})$-action). We succeed in associating to each diagram D a projective variety $\mathcal{F}_D$ and a line bundle whose global sections are isomorphic to S_D. These *configuration varieties* are generally singular, but reduce to an ordinary flag variety in the case when $D = \lambda$.

The key idea is that the processes of symmetrization and anti-symmetrization which define the Schur modules of a vector space $V = \mathbb{C}^n$ correspond to two geometric operations on the D-tuples of vectors V^D: symmetrization means restricting to the partial diagonal corresponding to the rows of D; and anti-symmetrization means taking the linear span of the subsets of vectors corresponding to the columns of D.

By desingularizing the configuration varieties with Bott-Samelson varieties, we obtain new fixed-point formulas for the polynomial characters s_D, which include the Schubert, skew Schur, and flagged Schur polynomials.

In subsequent work with Lakshmibai and Littelmann, we analyze the coordinate ring of a Bott-Samelson variety to obtain the crystal graph of the module S_D, along with a definition of generalized semi-standard Young tableaux. These tableaux are later characterized as the *peelable tableaux* of Reiner-Shimozono [RS], which are needed to prove the quiver polynomial formulas of Knutson-Miller-Shimozono [KMS].

2. Multiple flag varieties of finite type

The classical Schubert varieties are the B-orbits on a flag variety G/B, which correspond to the elements of the Weyl group of G. More generally we may consider the B-orbits on a product of partial flag varieties $G/P_1 \times$

$\cdots \times G/P_{k-1}$: the analogy is strongest when there are finitely many such orbits, so that the product is a *spherical variety.*

In this joint work with Andrei Zelevinsky and Jerzy Weyman, we explicitly classify such spherical varieties as well as their B-orbits in the case of the classical groups $G = \mathrm{GL}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$, and partially for $\mathrm{SO}_n(\mathbb{C})$. In fact we consider the equivalent problem of classifying the products $G/P_1 \times \cdots \times G/P_k$ with finitely many G-orbits.

The key idea is to translate the question into the language of quiver representations. A quiver Q is a directed graph, and $\mathrm{Rep}(Q)$ is the abelian category of its representations: vertices are represented by vector spaces and arrows by linear mappings. A partial flag consisting of p subspaces, each contained in the next, is naturally a representation of the directed path graph with p vertices. By the same token, a k-tuple of partial flags in a given vector space $\mathbb{C}^n$ is a representation of a *star quiver*, the union of k directed paths at a common endpoint.

Thus, the representations of a given star quiver Q contain the k-tuples of partial flags of every dimension, holding fixed only k and the number of subspaces in each partial flag. The G-orbits on these multiple flag varieties correspond to isomorphism classes of objects in $\mathrm{Rep}(Q)$.

Now the solution of our problem for $G = \mathrm{GL}_n(\mathbb{C})$ becomes a generalization of Gabriel's Theorem, which classifies those quivers having only a finite number of isomorphism classes in each dimension. Indeed, the simply-laced Dynkin diagrams of Gabriel's Theorem are star quivers, and appear in our classification, but we also get several new series since we ask only for finitely many isomorphism classes in a given dimension.

The classical theory of quivers deals only with the general linear groups, but surprisingly, we can adapt our technique to the other classical groups $G = \mathrm{Sp}_{2n}$ and $G = \mathrm{SO}_n$ by defining categories of symplectic and orthogonal representations of a star quiver, and relating these to the full representation category.

3. Bruhat order for two flags and a line

In the above classification of spherical multiple flag varieties for $G = \mathrm{GL}_n$, the simplest case after the flag variety G/B is the product $G/B \times \mathbb{P}^{n-1}$. Our paper makes a finer analysis of this case, characterizing the *Bruhat* or *closure order.* (This is equivalent to the geometry of G-orbits on the product $G/B \times G/B \times \mathbb{P}^{n-1}$, as in the paper's title.)

The analogous problem for G/B produces the classical Bruhat-Ehresmann order on the Weyl group $W = S_n$, described by its cover relations $w \lessdot r_{ij}w$ (where r_{ij} is a transposition) or alternatively by Ehresmann's tableau criterion: namely, for a flag $V_\bullet = (V_1 \subset \cdots \subset V_n) \in G/B$, the position of the orbit $B \cdot V_\bullet$ in Bruhat order is determined by its position with respect to the standard flag, namely by the rank numbers $\dim(V_i \cap \mathbb{C}^j)$ for all i, j.

We give explicit generalizations of these two descriptions to our case. We index orbits by permutation matrices specifying a flag $V_\bullet$, then "decorate" this matrix with a certain partition which specifies a line L. We define a series of moves on these decorated matrices which correspond to cover relations of the B-orbits. Next we give a tableau criterion, by which an element $(V_\bullet, L) \in G/B \times \mathbb{P}^{n-1}$ is measured by the previous rank numbers and by the new rank numbers $\dim((V_i + \mathbb{C}^j) \cap L)$.

One easily sees that the cover moves give an order weaker than the Bruhat order, while the rank numbers give a stronger order. To prove that these three orders are actually equivalent, we show that for any two elements related in the rank order, we can increase the smaller element by a cover move, while still maintaining the relation in rank order.

The combinatorial result is an interesting new ranked poset on decorated permutations which we conjecture to be lexicographically shellable, Sperner and unimodal (but not rank-symmetric).

4. Degeneracy schemes and Schubert varieties

Jointly with Lakshmibai, we exploit another connection between flags and quiver representations to answer a question of Buch-Fulton [BF] about representations of the oriented path quiver: namely, sequences of linear maps $V_1 \to \cdots \to V_n$.

Consider the variety of all such representations for some fixed complex vector spaces V_i. This variety is stratified by isomorphism classes in the category of quiver representations. The closure of each stratum is called a *degeneracy locus*: it allows one to associate a certain characteristic class to a sequence of maps of vector bundles $E_1 \to \cdots \to E_n$ over a fixed complex algebraic variety. Buch-Fulton gave explicit formulas for these characteristic classes in terms of the Chern classes of the bundles E_i, generalizing the Thom-Porteous formula.

These formulas become more geometrically useful if the degeneracy loci are Cohen-Macaulay varieties. We show this by exploiting a map of Zelevinksy which embeds our variety of quiver representations inside a certain partial flag variety G/P, where $G = \mathrm{GL}_N(\mathbb{C})$ with $N = \sum_i \dim(V_i)$. (This map is quite different from the trivial identification of a flag with a quiver representation.) In fact, this map takes each degeneracy locus to an open subset of a Schubert variety in G/B.

In our paper, we establish the scheme version of this picture. That is, a degeneracy locus is naturally defined as a (not necessarily reduced) determinantal scheme, and we show that Zelevinsky's map pulls back the known (reduced) scheme structure of a Schubert variety to this determinantal scheme. The proof requires only elementary linear algebra. Therefore the degeneracy scheme is indeed a reduced variety, and its singularities are no worse than those of the Schubert variety (which are known to be Cohen-Macaulay).

This technique is known as the *ubiquity of Schubert varieties*. Many determinantal varieties, such as ladder varieties (matrix rank varieties), the variety of (linear or circular) complexes, and the variety of nilpotent matrices can be analyzed by reducing them to Schubert varieties (possibly of a loop group). My Notes on Circular Complexes give a survey.

The result of our paper is another key ingredient in the Knutson-Miller-Shimozono analysis of quiver polynomials [KMS].

5. Littelmann paths for the basic representation of an affine Lie algebra

Our more recent work is inspired by the *product phenomenon*, a well-known but somewhat mysterious fact about the basic irreducible representations $V(\ell\Lambda_0)$ of an affine Lie algebra $\widehat{\mathfrak{g}}$ (or of a loop group). Namely, when these representations are restricted to the corresponding finite-dimensional Lie algebra $\mathfrak{g}$, they factor as semi-infinite tensor products of finite-dimensional $\mathfrak{g}$-modules. Furthermore the finite-dimensional Demazure module $V_{-\lambda}(\ell\Lambda_0)$, where $-\lambda$ is an anti-dominant translation in the affine Weyl group, factors as a $\mathfrak{g}$-module in many cases.

There are two main approaches to explaining this phenomenon. The first, due to the Kyoto school of Jimbo, Kashiwara, et. al., is in terms of the crystal graph of $V(\ell\Lambda_0)$, a kind of combinatorial skeleton of the $\widehat{\mathfrak{g}}$-module. The *Kyoto path model* constructs this crystal as an infinite tensor product of certain finite-dimensional $\widehat{\mathfrak{g}}$-crystals. These so-called *perfect crystals*, possessing delicate combinatorial properties, are conjectured to exist for all types, but the known constructions are fairly ad hoc, mainly for the classical types.

In this paper, we make a step toward a uniform construction of the Kyoto path model in the framework of Littelmann's path model, which realizes crystal graphs in terms of piecewise-linear paths in the vector space of weights. We generalize Littelmann's finite-length paths to certain semi-infinite paths called *skeins*. We then construct the Kyoto path model of the $\ell = 1$ basic representation $V(\Lambda_0)$ for those simple $\mathfrak{g}$ which possess a *minuscule coweight*. This includes the classical types as well as E_6, E_7. We also produce the path crystals of the Demazure modules $V_{-\lambda}(\Lambda_0)$.

A key ingredient is the automorphism of the extended Dynkin diagram (and hence of the affine Lie algebra) associated to each minuscule coweight.

Partly inspired by our work, Fourier-Littelmann [FL] recently gave an elegantly simple and general proof of the tensor product phenomenon (though not a construction of the crystal graph).

6. Product deformations of affine Schubert varieties

Our current work in progress involves a second explanation for the product phenomenon from the Borel-Weil perspective. The sum of Demazure modules $\bigoplus_{\ell\geq 0} V_\lambda(\ell\Lambda_0)$, with λ a fixed translation in the affine Weyl group, forms the projective coordinate ring of a (finite-dimensional) Schubert variety $\widehat{\mathrm{Gr}}_\lambda$ in the (infinite-dimensional) affine Grassmannian $\widehat{\mathrm{Gr}}$. Here $\widehat{\mathrm{Gr}} := G[t,t^{-1}]/G[t]$ is the quotient of the loop group $G[t,t^{-1}]$ by its parabolic subgroup $G[t]$.

The product formula $V_{-(\lambda+\mu)}(\ell\Lambda_0) \cong V_{-\lambda}(\ell\Lambda_0) \otimes V_{-\mu}(\ell\Lambda_0)$ would obviously follow if the affine Schubert variety $\widehat{\mathrm{Gr}}_{-(\lambda+\mu)}$ were isomorphic to the product variety $\widehat{\mathrm{Gr}}_{-\lambda} \times \widehat{\mathrm{Gr}}_{-\mu}$, but this is not true. However, Beilinson-Drinfeld have defined an algebraic family with special fiber $\widehat{\mathrm{Gr}}_{-(\lambda+\mu)}$ and general fiber $\widehat{\mathrm{Gr}}_{-\lambda} \times \widehat{\mathrm{Gr}}_{-\mu}$, meaning that the large Schubert variety can be deformed into the product (see [FBZ]). By examining Gaitsgory's [G] construction of the Beilinson-Drinfeld Grassmannian, we have proved that the family is flat, which implies the product formula for Demazure modules.

Furthermore, over the complex numbers $\widehat{\mathrm{Gr}}$ is homeomorphic to ΩK, the group of based loops in the maximal compact subgroup $K \subset G$; and the Schubert variety $\widehat{\mathrm{Gr}}_{-(\lambda+\mu)}$ is homeomorphic to a correponding subspace $\Omega K_{-(\lambda+\mu)} \subset \Omega K$.

We prove a topological version of the product phenomenon: Suppose $\Omega K_{-\mu}$ is a K-space. Then $\Omega K_{-(\lambda+\mu)}$ has a Bott-Samelson resolution $\Omega K_{(-\lambda,-\mu)}$ which is homeomorphic to the product space:

$$\Omega K_{-\lambda} \times \Omega K_{-\mu} \cong \Omega K_{(-\lambda,-\mu)} \xrightarrow{\text{birat}} \Omega K_{-(\lambda+\mu)}\,.$$

Therefore, the Beilinson-Drinfeld deformation can be lifted to family of complex structures on a fixed topological space.

7. Schubert varieties of a loop group

In other current work, joint with Mark Shimozono, we examine the homology ring $H^*(\Omega K)$, where multiplication is induced by pointwise product of loops. Bott [B] gave an explicit presentation of this *Pontryagin ring* as a quotient of a polynomial ring, and showed that it has a linear basis of Schubert cycles ΩK_λ. We have succeeded in giving an explicit divided-difference formula for the *affine homology Schubert polynomials* $\mathfrak{S}_\lambda$, the representive of ΩK_λ in this presentation, for any semi-simple K. Thanks to the factorization phenomenon above, it is sufficient to determine $\mathfrak{S}_\lambda$ for a finite number of "prime" classes.

We are presently trying to prove Shimozono's conjecture that for type A, our homology Schubert polynomials coincide with the dual k-Schur polynomials of Lapointe-Lascoux-Morse [LLM].

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