WHAT IS...

an Expander?

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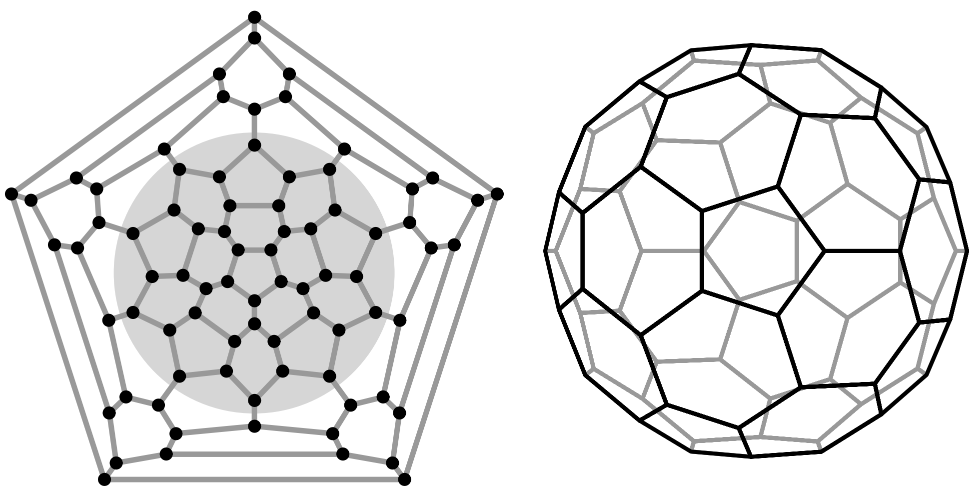

This Ramanujan graph has 80 vertices,which is close to the largest known planar Ramanujan graph of 84 vertices. Its girth is 5, its expansion constant 1/4 (as indicated by the shaded circle), and λ_1 has been calculated by A. Gamburd to be 2.81811... It may be constructed by shrinking pentagons on a dodecahedron.

By a graph $X = (V, E)$ we mean a finite set V of vertices and a set E of pairs of these vertices called edges. In words, an expander is a highly connected sparse graph X. It is this apparently contradictory feature of being both highly connected and at the same time sparse that on the one hand makes the existence of such graphs counterintuitive and on the other hand makes them so useful. Their uses in pure mathematics include, for example, in combinatorics the explicit construction of graphs with large girth (this being the length of the shortest nonbacktracking closed circuit) and large chromatic number (this being the least number of colors needed to paint the vertices so that adjacent vertices have distinct colors), and in functional analysis the construction of finitely generated groups which cannot be embedded uniformly in a Hilbert space (Gromov) and related counterexamples to the Baum-Connes conjectures for group actions. However, it is in applications in theoretical computer science where expanders have had their major impact. Among their applications are the design of explicit superefficient communication networks, constructions of error-correcting codes with very efficient encoding and decoding algorithms, derandomization of random algorithms, and analysis of algorithms in computational group theory (see for example [R-V-W] and the references therein).

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The formal definition of the most basic expander is as follows: Let $k \geq 2$ be an integer, and let X be a k-regular graph (that is, each vertex $v \in V$ has exactly k neighbors). The Cheeger constant h of X is defined to be

$$h(X) = \min_{\emptyset \underset{\neq}{\subset} F \underset{\neq}{\subset} V} \frac{|\partial F|}{\min(|F|, |V \backslash F|)},$$

where ∂F is the set of edges running from F to its complement $V \backslash F$ and $|\partial F|$ is its cardinality. X is also called an $[n, k, h]$ expander where $|X| = n$. Note that $h > 0$ if and only if X is connected. Also, if $|F| \leq \frac{n}{2}$, then $|\partial F| \geq h|F|$, so that if h is not small, then every such subset F has many neighbors outside F (hence the name expander). By expanders we really have in mind a sequence of such $[n, k, h]$'s with k and $\varepsilon_0 > 0$ fixed, $h \geq \varepsilon_0$, and $n \to \infty$. It is easy to see that for this to happen k must be at least 3, which we assume henceforth. That $h \geq \varepsilon_0 > 0$ ensures that the graphs are highly connected, while k being fixed (and $|E| = \frac{kn}{2}$) ensures that they are sparse.

It is perhaps surprising that expanders exist. The first proof of their existence by Pinsker is based on counting arguments. Consider the probability space of all $X_{n,k}$'s (that is, k-regular graphs on n marked vertices) where each such $X_{n,k}$ is chosen with equal probability. Then for $k \geq 3$ there is $\varepsilon(k) > 0$ such that the probability that $h(X_{n,k}) \geq \varepsilon$ tends to 1 as $n \to \infty$.

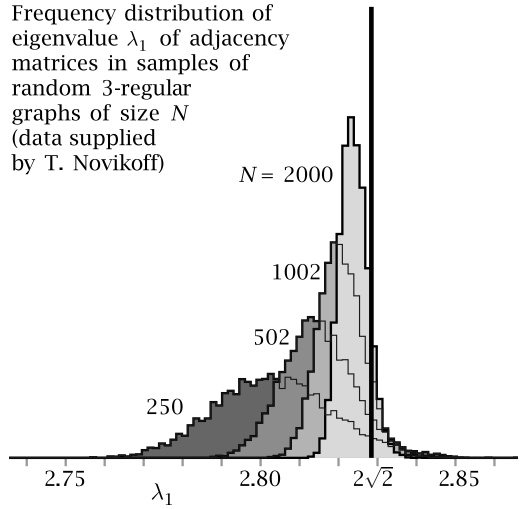
For applications one wants explicit constructions. A useful means to achieve this is the following spectral method. Given an $X = X_{n,k}$, let A be the $n \times n$ symmetric matrix whose rows and columns are indexed by the vertices $v \in V$ and for which the v, w entry is 1 if v is joined to w, and 0 otherwise. The vector $(1, 1, \ldots, 1)$ is an eigenvector of A with eigenvalue k. The eigenvalues of A are real and lie in the interval $[-k, k]$. Let $\lambda_1(X)$ denote the next-to-largest eigenvalue of A after k. The following inequalities, which are the discrete analogues of inequalities of Cheeger and Buser in differential geometry, relate $h(X)$ to the gap between $\lambda_1(X)$ and k and are due to Tanner, Alon, and Milman:

$$\frac{k - \lambda_1(X)}{2} \leq h(X) \leq \sqrt{2k(k - \lambda_1(X))}.$$

Thus $X_{n,k}$ is an expanding family if and only if it has a uniform lower bound on the spectral gap, and the larger the gap the better the expansion. Using this and quotients of explicit infinite discrete groups which enjoy Kazhadan's property T (which is the property that the trivial representation is isolated in the space of all unitary representations), Margulis gave the first examples of explicit expanders. There is a limit as to how big the spectral gap can possibly be (Alon-Boppana): For k fixed

$$\liminf_{n \to \infty} \lambda_1(X_{n,k}) \geq 2\sqrt{k-1}.$$

The liminf is taken over all $X_{n,k}$'s. An $X_{n,k}$ (or rather a sequence of such $X_{n,k}$'s with $n \to \infty$, k fixed) is called a Ramanujan graph if $\lambda_1(X_{n,k}) \leq 2\sqrt{k-1}$. In view of the above, such a graph is optimal, at least as far as the spectral gap measure of expansion is concerned. It is a pleasant fact that if $k = q + 1$ with q a prime power, then Ramanujan graphs exist. The constructions are in terms of very explicit descriptions as Cayley graphs of $PGL(2, \mathbb{F}_q)$ [L-P-S], [M] (Cayley graphs are those whose vertex sets are the elements of a group G and whose edges correspond to $g \to gg_1$ with g_1 in a set of generators). For k not of this form it is an interesting question as to whether Ramanujan graphs exist. J. Friedman has recently shown that for k fixed and $\varepsilon > 0$ the probability that $\lambda_1(X_{n,k}) \leq 2\sqrt{k-1} + \varepsilon$ tends to 1 as $n \to \infty$. So the random graph is asymptotically Ramanujan. Some very interesting numerical experimentation (T. Novikoff, http://www.math.nyu.edu/Courses/V63.0393/, honors math lab, projects) indicates that the probability that the random graph is Ramanujan is slightly bigger than 0.5, corresponding to the skewness in the Tracy-Widom distribution in random matrix theory (specifically the "Gaussian Orthogonal Ensemble").



Frequency distribution of eigenvalue λ_1 of adjacency matrices in samples of random 3-regular graphs of size N (data supplied by T. Novikoff)

From http://www.math.nyu.edu/Courses/V63.0393/(Projects)

In [R-V-W] it is shown how one may use their novel "zig-zag graph product" (a notion related to the semidirect product of groups when the graphs are Cayley graphs) to give new explicit constructions of expanders. Their construction of the family is inductive, with the zig-zag product being used at each step. It has the extra flexibility that allows them to construct explicit expanders that expand small sets almost optimally (they call these "lossless expanders"). This extra twist turns out to be very useful in a number of applications, and this feature cannot be deduced from a spectral gap analysis.

References

[L-P-S] A. Lubotzky, R. Phillips, and P. Sarnak, *Combinatorica* **8** (1988), 261–277.

[M] G. Margulis, *Problems Inform. Transmission* **24** (1988), 39–46.

[R-V-W] O. Reingold, S. Vadhan, and A. Wigderson, *Ann. of Math.* **155** (2002), 157–187.