Math 881 **HW 3/17** **Spring 2006**

We consider a probability space of k-regular bipartite multi-graphs on $n+n$ vertices $V = V_+ \cup V_-$. We construct such a graph from $\pi_1, \ldots, \pi_k \in \mathrm{Perm}(n)$, permutations of $\{1, \ldots, n\}$ chosen independently with uniform probablity. The corresponding multi-graph $G'' := G''(\pi_1, \ldots, \pi_k)$ is defined by the neighbor sets $N(i_+) := \{\pi_1(i)_-, \ldots, \pi_k(i)_-\}$.

Proposition For $k \geq 7$, the graph G'' is a $\frac{1}{2}$-expander with probability 1:

$$\mathbb{P}\big(\, c'(G'') \geq \tfrac{1}{2} \,\big) \to 1 \quad \text{as} \quad n \to \infty\,,$$

where c' means the bipartite expander constant. That is, for almost any G'' chosen as above, and any $S \subset V_-$ with $|S| \leq \frac{n}{2}$, we have $|N(S)| \geq (1+\frac{1}{2})|S|$.

Proof: We aim to show:

$$\mathbb{P}\big(\, c'(G'') < \tfrac{1}{2} \,\big) \overset{??}{\to} 0 \quad \text{as} \quad n \to \infty\,.$$

Now, the graph G'' fails to be a $\frac{1}{2}$-expander if there exists $S \subset V_+$ and $T \subset V_-$ such that:

$$|S| = s \leq \tfrac{1}{2}n \quad , \quad |T| = \lfloor \tfrac{3}{2}s \rfloor \quad , \quad N(S) \subset T\,.$$

The last condition means precisely that $\pi_1(S), \ldots, \pi_k(S) \subset T$.

Fixing S, T, we compute the probability of uniformly choosing a permutation π with $\pi(S) \subset T$:

$$\mathbb{P}(\, \pi(S) \subset T\,) = \frac{t(t{-}1)\cdots(t{-}s{+}1)\cdot(n{-}s)!}{n!} = \frac{t!\,(n{-}s)!}{s!\,n!}\,,$$

where $t := \lfloor \frac{3}{2}s \rfloor$. Since $\pi_1, \ldots, \pi_k$ are chosen independently, we have:

$$\mathbb{P}(\, \pi_1(S), \ldots, \pi_k(S) \subset T\,) = \left[\frac{t!\,(n{-}s)!}{s!\,n!}\right]^k .$$

Letting S, T run over all possible subsets gives:

$$\begin{aligned} \mathbb{P}\big(\, c'(G'') < \tfrac{1}{2} \,\big) &= \mathbb{P}(\, \exists\, S, T \text{ s.t. } \pi_1(S), \ldots, \pi_k(S) \subset T\,) \\ &\leq \sum_{s=1}^{n/2} \binom{n}{s} \binom{n}{t} \left[\frac{t!\,(n{-}s)!}{s!\,n!}\right]^k . \end{aligned}$$

Denoting the summand as $R(s)$, we thus have:

$$\mathbb{P}\big(\, c'(G'') < \tfrac{1}{2} \,\big) \;\leq\; \sum_{s=1}^{n/3} R(s) \;+\; \sum_{s=n/3}^{n/2} R(s)\,,$$

and we must show that each of the above summations tends to zero.

Consider the summation over $s \le \frac{1}{3}n$. For an even value of s we have $\lfloor \frac{3}{2}(s+1) \rfloor = \frac{3}{2}s + 1$ and:

$$\begin{aligned} \frac{R(s)}{R(s+1)} &= \frac{s+1}{n-\frac{3}{2}s} \left[\frac{n-s}{\frac{3}{2}s+1} \right]^{k-1} \\ &= \frac{s+1}{\frac{3}{2}s+1} \cdot \frac{n-s}{n-\frac{3}{2}s} \cdot \left[\frac{n-s}{\frac{3}{2}s+1} \right]^{k-2} \\ &\ge \frac{2}{3} \cdot 1 \cdot \left[\frac{4}{3} \right]^{k-2} \ \ge \ \frac{2}{3} \left[\frac{4}{3} \right]^{2} \ > \ 1 \,, \end{aligned}$$

since $f(s) := (n-s)/(\frac{3}{2}s+1)$ is increasing for fixed n and $s \le \frac{1}{3}n$. (Also recall $k \ge 4$.) We can make a similar calculation for odd values of s. Hence $R(s)$ is decreasing over the interval $1 \le s \le \frac{1}{3}n$, and:

$$\sum_{s=1}^{n/3} R(s) \ \le \ \tfrac{1}{3}n \, R(1) \ = \ \tfrac{1}{3} n^{3-k} \ \to \ 0 \,,$$

since $k \ge 4$.

Finally, consider the summation over the interval $\frac{1}{3}n \le s \le \frac{1}{2}n$. For large n, the value s is also large, so we may approximate $R(s)$ using Stirling's formula:

$$n! \ \sim \ \sqrt{2\pi n} \left(\frac{n}{e} \right)^{n} ,$$

in which the percentage error approaches zero as $n \to \infty$. For $0 < \alpha < 1$ and $\beta := 1 - \alpha$, we compute:

$$\binom{n}{\alpha n} \sim \frac{1}{\sqrt{2\pi n}} \ \alpha^{-\alpha n - \frac{1}{2}} \ \beta^{-\beta n - \frac{1}{2}} \,,$$

and

$$\binom{n}{s} \binom{n}{t} \le \binom{n}{\frac{1}{2}n}^{2} \sim \frac{2^{2n}}{8\pi n}$$

Furthermore, for fixed n and $s = \alpha n$,

$$Q(s) := \frac{t! \, (n-s)!}{s! \, n!} \ \sim \ 3^{\frac{1+3\alpha n}{2}} \ \beta^{\beta n + \frac{1}{2}} \ \left(\tfrac{1}{2}\alpha\right)^{\alpha n} ,$$

$$\log Q(s) = \alpha n \, \log \left(\frac{3\sqrt{3}}{2} \, \frac{\alpha}{1-\alpha} \right) + 1/2 \log(1-\alpha) + \log \sqrt{3} \,,$$

$$\frac{d}{d\alpha} \log Q(s) = n \left(\log \left(\frac{3\sqrt{3}}{2} \, \frac{\alpha}{1-\alpha} \right) + \frac{1}{1-\alpha} \right) - \frac{1}{2(1-\alpha)} \,,$$

and it is easily seen that this is positive for large n and $\frac{1}{3} \le \alpha \le \frac{1}{2}$. Hence $\log Q(s)$ is increasing, and so is $Q(s)$. Thus:

$$\begin{aligned}
\textstyle\sum_{s=n/3}^{n/2} R(s) &\le \sum_{s=n/3}^{n/2} \frac{2^{2n}}{8\pi n}\, Q(s)^k \\
&\le \tfrac{1}{6} n\, \frac{2^{2n}}{8\pi n}\, Q(\tfrac{n}{2}) \\
&\sim c\, 4^n \left(\tfrac{3}{4}\right)^{\frac{3}{4}kn} \\
&= c\, \left(\frac{3^{3k/4}}{4^{3k/4-1}}\right)^n
\end{aligned}$$

The last quantity in parentheses is < 1 only for $k \ge 7$. Perhaps there is a better estimate that works for $k \ge 4$?