Proposition *(Hall Matching) Let $G = (A \cup B, E)$ be a bipartite graph satisfying Hall's Condition: for any subset $A' \subset A$, we have $|A'| \leq |N(A')|$. Then there exists a matching from A to B.*

Here $N(A') \subset B$ denotes the neighbor set of A'. A *matching* from A to B means a set of edges $M \subset E$ such that every $a \in A$ lies on exactly one edge of M, and every $b \in B$ lies on at most one edge of M.

Proof: We first convert G into a flow network $\hat{G}$ whose directed graph includes G with all edges oriented from A to B; an extra source vertex s with edges $s \to a$ for all $a \in A$; and an extra sink vertex t with edges $b \to t$ for all $b \in B$. Also we set the capacity of each edge to $c = 1$.

A partial matching M' from $A' \subset A$ to B clearly corresponds to a flow f in $\hat{G}$: to each edge $(a \to b) \in M'$ there corresponds $f(s,a) = f(a,b) = f(b,t) = 1$, with all other $f(x,y) = 0$. Furthermore, $|A'| = \text{Flow}(f)$. Thus a matching of a maximal subset $A' \subset A$ corresponds to a maximal flow in $\hat{G}$.

Now, by the Max-Flow/Min-Cut Theorem, this is the same as the minimum capacity of a cut (S,T):

$$\min_{(S,T)} \text{Cap}(S,T) = \max_{A'} |A'| \, .$$

(Recall that a pair of disjoint vertex sets (S,T) is a cut if $s \in S,\ t \in T$ and $S \cup T = A \cup B \cup \{s,t\}$. Also, $\text{Cap}(S,T) := |E(S,T)|$, the number of directed edges from S to T.) Thus, a complete matching $(A' = A)$ is possible for G exactly when the minimal capacity for a cut of $\hat{G}$ is $n := |A|$.

Let (S,T) be any cut, and denote $S_A := S \cap A$, etc., so that $A = S_A \cup T_A$ and $B = S_B \cup T_B$. The capacity is:

$$\text{Cap}(S,T) = |T_A| + |S_B| + |E(S_A\,, T_B)| \, .$$

Now, $N(S_A) \cap S \subset S_B$, so $|S_B| \geq |N(S_A) \cap S|$. Also, every vertex of $N(S_A) \cap T$ corresponds to at least one edge in $E(S_A, T_B)$, so $|E(S_A,\ , T_B)| \geq |N(S_A) \cap T|$. Thus we have:

$$\begin{aligned} \text{Cap}(S,T) &\geq |T_A| + |N(S_A) \cap S| + |N(S_A) \cap T| \\ &= |T_A| + |N(S_A)| \, . \end{aligned}$$

Applying Hall's Condition $|N(S_A)| \geq |S_A|$ gives:

$$\text{Cap}(S,T) \geq |T_A| + |S_A| = |A| = n \, .$$

Thus any cut has capacity $\geq n$, and a minimal cut has capacity exactly n, as desired. $\square$

Here are some auxilliary results that some people used in their proofs.

Claim 1: Suppose a cut (S,T) of $\hat{G}$ has an edge $a \rightarrow b$ with $a \in S_A$, $b \in T_B$; then $(S \cup \{a\}, T - \{b\})$ is a cut with smaller or equal capacity.

Proof:

$$\begin{aligned} \mathrm{Cap}(S,T) &= |T_A| + |S_B| + |E(S_A\,,T_B)|\,, \\ \mathrm{Cap}(S \cup \{a\}, T - \{b\}) &= |T_A| + |S_B| + 1 + |E(S_A\,,T_B)| - |E(S_A\,,b)| \\ &\leq \mathrm{Cap}(S,T)\,, \end{aligned}$$

since $|E(S_A\,,b)| \geq 1$. □

We can apply this fact repeatedly to find a minimal cut with $E(S_A, T_B) = \emptyset$, which simplifies the capacity formula to: $\mathrm{Cap}(S,T) = |S_A| + |T_B|$.

Claim 2: For a general bipartite graph, the Augmenting Algorithm produces a cut (S,T) with $E(T_A, S_B) = \emptyset$.

FALSE! Here is a counterexample: Let $|A| = 3$, $|B| = 2$, and

$$E = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_3, b_2)\}\,.$$

Take the max flow corresponding to the matching $M = \{(a_1, b_1), (a_2, b_2)\}$. Then the corresponding cut is:

$$S = \{s, a_2, a_3, b_2\} \quad , \quad T = \{a_1, b_1, t\}\,.$$

Nevertheless, $(a_1, b_2) \in E(T_A, S_B)$. □