Math 881 HW 2/17 Solutions Spring 2006

Super-concentrators We seek to construct graphs $S(n)$ with n input and n output vertices $I, O \subset V$, such that for any k and any $A \subset I$ and $B \subset O$ with $|A| = |B| = k$, there exist k vertex-disjoint paths in $S(n)$ starting in A and ending in B.

Proposition: Let $S_1(n) = (V, E)$ with

$$V = I \cup O = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\},$$

$$E = \{a_i b_i\}_{1 \le i \le n} \ \cup \ \{b_i b_j\}_{1 \le i < j \le n}.$$

Then $S_1(n)$ is an undirected super-concentrator with $\frac{1}{2}n(n+1)$ edges.

Proof. Denote $a_i' = b_i$ and $b_i' = a_i$. Let $A_0 = A \setminus (A \cap B')$ and $B_0 = B \setminus (B \cap A')$. Then take $A \cap B'$ to $B \cap A'$ along the a—b edges. Also take A_0 to A_0' along the a—b edges, then take A_0' to the disjoint set B_0' by an arbitrary matching in the complete graph O.

Conjecture: Let $S_2(n) = (V, E)$ with:

$$V = I \cup O = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\},$$

$$E = \{a_i b_{i+j} \mid \text{ where } 0 \le j \le \frac{1}{2}(n-1)\}.$$

where we consider the subscript $i + j$ modulo n. Then $S_2(n)$ is an undirected super-concentrator with $\frac{1}{2}n^2$ edges for n even, and $\frac{1}{2}n(n+1)$ edges for n odd.

Concentrators We also seek graphs $C(n, m)$ with n input and m output vertices $I, O \subset V$, such that for any $k \le n/2$ and any $A \subset I$ with $|A| = k$, there exist k vertex-disjoint paths in $C(n, m)$ starting in A and ending in O.

Proposition: Let $C_1(n, m) = (V, E)$ with:

$$V = I \cup O = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\},$$

$$E = \{a_i b_i\}_{1 \le i \le m} \ \cup \ \{a_i b_1, \ldots, a_i b_{\lceil n/2 \rceil}\}_{m+1 \le i \le n}.$$

Then $C_1(n, m)$ is a concentrator with $\lceil \frac{n}{2} \rceil (n - m) + m$ edges.