Math 880 **The Twelvefold Way** **Spring 2022**

We count the number of possible functions f with input set $[k]=\{1,2,\ldots,k\}$ and output set $[n]=\{1,2,\ldots,n\}$, subject to restrictions (injective or surjective). We may picture f as a way of distributing k balls (marked $1,\ldots,k$) into n baskets (marked $1,\ldots,n$). A map is injective if each basket contains at most one ball, or surjective if no basket is empty.

Indistinguishable $[k]$ means we consider two functions the same whenever they differ by a permutation of the inputs $[k]$; so we picture the k balls as identical, unmarked. Similarly, *indistinguishable* $[n]$ means we consider classes of functions up to permutation of the outputs $[n]$, so we picture the n baskets as identical and movable, and we cannot distinguish a first basket, second basket, etc.

$f\ :\ [k]\ \to\ [n]$	ALL FUNCTIONS	INJECTIONS $(k \le n)$	SURJECTIONS $(k \ge n)$
DIST DIST	① n^k	② $n^{\underline{k}}$	③ $\mathrm{surj}(k,n) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^k$
IND DIST	④ $\left(\!\binom{n}{k}\!\right) = \frac{n^{\overline{k}}}{k!}$ $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right)$	⑤ $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$ $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$	⑥ $a(k,n) = \left(\!\binom{n}{k-n}\!\right) = \binom{k-1}{n-1}$
DIST IND	⑦ $\left\{k \atop \le n\right\} = \left\{k \atop 1\right\} + \left\{k \atop 2\right\} + \cdots + \left\{k \atop n\right\}$	⑧ 1	⑨ $\left\{k \atop n\right\} = \frac{\mathrm{surj}(k,n)}{n!}$
IND IND	⑩ $p_{\le n}(k) = p_1(k) + p_2(k) + \cdots + p_n(k)$	⑪ 1	⑫ $p_n(k) = p_{\le n}(k-n)$

- Binomial coefficient, choose number $\binom{n}{k}$: $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. Multiset, multi-choose number $\left(\!\binom{n}{k}\!\right)$: $\frac{1}{(1-x)^n} = \sum_{k\ge0}\left(\!\binom{n}{k}\!\right)x^k$, $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$.
- Stirling partition number (second kind) $\left\{n \atop k\right\}$: $x^n = \sum_{k=1}^{n}\left\{n \atop k\right\}x^{\underline{k}}$, $x^{\underline{n}} = \sum_{k=1}^{n}(-1)^{n-k}\left[n \atop k\right]x^k$, $\sum_{n\ge0}\left\{n \atop k\right\}\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k$.
- Stirling cycle number (first kind) $\left[n \atop k\right]$: counts permutations $w \in S_n$ with k cycles, $\left[n \atop k\right] = \left[n-1 \atop k-1\right] + (n-1)\left[n-1 \atop k\right]$, $\sum_{n\ge0}\left[n \atop k\right]\frac{x^n}{n!} = \log^k(\frac{1}{1-x})$.
- Bell number $B_k = \left\{k \atop \le k\right\} = \left\{k \atop 1\right\} + \left\{k \atop 2\right\} + \cdots + \left\{k \atop k\right\}$. Dobinski: $B_k = \frac{1}{e}\sum_{i\ge0}\frac{i^k}{i!}$. $\sum_{k\ge0}B_k\frac{x^k}{k!} = e^{(e^x-1)}$. Recurrence: $B_k = \sum_{i=0}^{k-1}\binom{k-1}{i}B_i$.
- Partition number $p(k) = p_{\le k}(k) = p_1(k) + p_2(k) + \cdots + p_k(k)$: $\sum_{k\ge0}p(k)x^k = \prod_{i\ge1}\frac{1}{1-x^i}$, Hardy-Ramanujan: $p(k) \sim \frac{1}{4n\sqrt{3}}\exp(\pi\sqrt{\frac{2n}{3}})$.
- Fibonacci number $F_k = F_{k-1} + F_{k-2}$ from $F_0 = 0$, $F_1 = 1$. Binet: $F_k = \frac{1}{\sqrt{5}}(\phi^k - (-\psi)^k) = \mathrm{round}(\frac{\phi^k}{\sqrt{5}})$, where $\phi = \frac{\sqrt{5}+1}{2}$, $\psi = \frac{\sqrt{5}-1}{2}$.
- Catalan number $C_k = \sum_{i=0}^{k-1}C_iC_{k-i}$ from $C_0 = 1$; $C_k = \frac{1}{k+1}\binom{2k}{k}$. Counts: binary ordered trees (k+1 leaves); ordered trees (k nodes).
- Derangement number (perms with no fixed points) $D_n = n!(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!})$. $\sum_{k\ge0}D_k\frac{x^k}{k!} = \frac{e^{-x}}{1-x}$. $D_n = (n-1)(D_{n-1}+D_{n-2})$.
- Cayley: labeled, unordered trees $T_n = n^{n-2}$. Unlabeled, unordered rooted trees r_n: $\sum_{n\ge1}r_nx^n = x\prod_{i\ge1}\frac{1}{(1-x^i)^{r_i}}$; $r_{n+1} = \frac{1}{n}\sum_{k=1}^{n}\sum_{j|k}jr_jr_{n-k+1}$.