Following Flajolet-Sedgewick Ch. II, a labeled graded combinatorial class $\tilde{\mathcal{A}} = \coprod_{k\geq 0} \tilde{\mathcal{A}}_k$ comprises elements $a \in \tilde{\mathcal{A}}_k$ each having k atoms labeled with a permutation of $[k] = \{1, 2, \ldots, k\}$. For a set $S = \{s_1 < \cdots < s_k\}$, we define a_S to be a with each atom label i replaced with s_i.

For labeled classes $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$, the *labeled product* is the labeled class:

$$\tilde{\mathcal{C}} \;=\; \tilde{\mathcal{A}} \star \tilde{\mathcal{B}} \;=\; \left\{ (a_S, b_T) \;\middle|\; \begin{array}{c} a \in \tilde{\mathcal{A}}_i,\ b \in \tilde{\mathcal{B}}_j \\ |S| = i,\ |T| = j \\ S \sqcup T = [\,i{+}j\,] \end{array} \right\}.$$

In the ordered pair (a_S, b_T), the label set $[i{+}j]$ is distributed among the atoms of a and b. As usual, the size function is $|(a_S, b_T)| = |a| + |b|$. Denoting the counting sequence $C_k = \#\tilde{\mathcal{C}}_k$, etc., we get:

$$C_k \;=\; \sum_{i=0}^{k} \binom{k}{i} A_i B_{k-i} \;=\; k! \sum_{i=0}^{n} \frac{A_i}{i!}\, \frac{B_{k-i}}{(k{-}i)!}\,,$$

and the generating function:

$$\tilde{C}(x) \;=\; \sum_{k\geq 0} \frac{C_k}{k!} x^k \;=\; \sum_{k\geq 0} \sum_{i=0}^{k} \frac{A_i}{i!}\, \frac{B_{k-i}}{(k{-}i)!}\, x^k \;=\; \tilde{A}(x)\tilde{B}(x).$$

Next we define the *directed labeled product* [FS II.6.3, p. 139]:

$$\tilde{\mathcal{D}} = \tilde{\mathcal{A}}^{\min} \star \tilde{\mathcal{B}} = \{(a_S, b_T) \in \tilde{\mathcal{A}} \star \tilde{\mathcal{B}} \mid 1 \in S\},$$

restricting to those relabelings where the minimum label 1 appears in the first component a_S. (We can analogously define $\tilde{\mathcal{A}} \star \tilde{\mathcal{B}}^{\min}$, $\tilde{\mathcal{A}}^{\max} \star \tilde{\mathcal{B}}$, etc.) The number of ways to choose $S \sqcup T = [k]$ with $|S| = i$ and $1 \in S$ is $\binom{k-1}{i-1}$, so we have the counting sequence:

$$D_k \;=\; \sum_{i=0}^{k} \binom{k{-}1}{i{-}1} A_i B_{k-i} \;=\; \sum_{i=0}^{k} \frac{i}{k} \binom{k}{i} A_i B_{k-i} \;=\; \frac{k!}{i} \sum_{i=0}^{k} \frac{iA_i}{i!}\, \frac{B_{k-i}}{(k{-}i)!}\,.$$

Recall the formal derivative and integral operations (inverses of each other) on $F(x) = \sum_{k\geq 0} F_k x^k \in \mathbb{C}[\![x]\!]$:

$$xF'(x) = \sum_{k\geq 0} kF_k x^k \quad , \quad \int \frac{1}{x} F(x)\, dx = \sum_{k\geq 1} \frac{1}{k} F_k x^k.$$

Then we get the generating function:

$$x\tilde{D}'(x) \;=\; \sum_{k\geq 0} k \frac{D_k}{k!} x^k \;=\; \sum_{k\geq 0} \sum_{i=0}^{k} \frac{iA_i}{i!}\, \frac{B_{k-i}}{(k{-}i)!}\, x^k \;=\; x\tilde{A}'(x)\, \tilde{B}(x),$$

and:

$$\tilde{D}(x) \;=\; \int \frac{1}{x}\, x\tilde{A}'(x)\, \tilde{B}(x)\, dx \;=\; \int \tilde{A}'(z)\, \tilde{B}(z)\, dz.$$

Similarly the generating function of $\tilde{\mathcal{A}} \star \tilde{\mathcal{B}}^{\min}$ is $\int \tilde{A}(x)\, \tilde{B}'(x)\, dx$, and the integration by parts formula:

$$\int \tilde{A}'(x)\, \tilde{B}(x)\, dx \;=\; \tilde{A}(x)\tilde{B}(x) - \int \tilde{A}(x)\, \tilde{B}'(x)\, dx$$

can be interpreted combinatorially as saying the label 1 is in the first component exactly when it is not in the second component.

A construction like $[1]^{\min} \star \tilde{\mathcal{B}}$ is useful for describing a Deletion Transform in which we remove the atom with the minimum label 1 (or similarly, the maximum label k).