Math 880 **Homework 7** **Fall 2025**

Give credit for significant help from a reference or person.

1. Consider the ring of symmetric polynomials in $n = 3$ variables,

$$\Lambda^{(3)} = \mathbb{Q}[x_1, x_2, x_3]^{S_3},$$

with its bases m_λ (monomial), e_λ (elementary), h_λ (homogeneous), p_λ (power), indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$. The degree $k = 3$ component $\Lambda_k^{(n)} = \Lambda_3^{(3)}$ has dimension $p(k) = 3$ indexed by the partitions $\lambda = (3), (2,1), (1,1,1)$.

a. By hand, write each basis of $\Lambda_3^{(3)}$ in terms of x_1, x_2, x_3.

b. Write e_λ, h_λ, p_λ in terms of the monomial basis m_λ, and verify the change-of-basis coefficients for the first two denoted in Stanley as $M_{\lambda\mu}$ (0-1 matrices) and $N_{\lambda\mu}$ ($\mathbb{N}$-matrices). Which of these bases has a triangular relationship to m_λ, for an appropriate ordering of rows and columns?

c. Recall the involution ω on $\Lambda_3^{(3)}$ defined by $\omega(e_\lambda) = h_\lambda$. Compute $[\omega]_m^m$, the 3×3 matrix of the linear mapping ω with respect to the monomial basis $m = \{m_\lambda\}$. Square the matrix to verify it is an involution. Verify that p_λ are eigenvectors with eigenvalues $\epsilon(\lambda) = \mathrm{sgn}(w_\lambda)$, where w_λ is a permutation with cycle type λ.

d. Inner product. Stanley defines the inner product (symmetric, bilnear, non-degenerate) on Λ by $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$: that is, the monomial and homogeneous bases are orthogonal by definition. Find the 3×3 matrix J of the inner product with respect to the monomial basis of $\Lambda_3^{(3)}$, so that for $f, g \in \Lambda_3^{(3)}$ corresponding to 3×1 coordinate vectors $[f]_m, [g]_m \in \mathbb{C}^3$, we have:

$$\langle f, g\rangle = {}^t[f]_m \cdot J \cdot [g]_m$$

where the right side is a matrix product $(1 \times 3)\cdot(3\times 3)\cdot(3\times 1) = (1\times 1)$.
Verify that p_λ is an orthogonal, but not orthonormal, basis of $\Lambda_3^{(3)}$. (In fact, $\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda\mu}$, where $z_\lambda = \#N_{\mathfrak{S}_3}(w_\lambda)$ the number of permutations in $\mathfrak{S}_3$ which commute with a given permutation w_λ of cycle type λ.)

2. Schur polynomials. Keeping $n = 3$ variables, we defined $s_\lambda(x_1, x_2, x_3) = \sum_T x^T$ over $T \in \mathrm{SSYT}_3(\lambda)$, the semi-standard Young tableaux T with shape λ and entries $T(i,j) \leq 3$. We expand $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ with change of basis coefficients given by the Kostka numbers $K_{\lambda\mu}$, the number of SSYT of shape λ and content μ (i.e. T contains μ_i entries equal to i).

a. Enumerate the SSYT for all $|\lambda| = 3$, and write out the Schur polynomials $s_\lambda \in \Lambda_3^{(3)}$ in terms of the monomial basis.

b. For each $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ above, construct the irreducible representation V_λ of $\mathrm{GL}_3(\mathbb{C})$ inside the ring of polynomial functions on 3×3 matrices $X = (x_{ij})$, the ring $R = \mathbb{C}[X] = \mathbb{C}[x_{ij}]$ for $i, j \in [3]$. Define V_λ as the $\mathbb{C}$-span of the polynomials

$$\Delta^{I_1}_{[\lambda'_1]}(X)\,\Delta^{I_2}_{[\lambda'_2]}(X)\,\Delta^{I_3}_{[\lambda'_3]}(X)\,,$$

where $\Delta^I_J(X)$ is a minor of the matrix $X = (x_{ij})$, the determinant of the submatrix on rows I and columns J. Here the row-sets I_j are arbitrary, but all columns are left-justified: $J_i = [\lambda'_i] = \{1, \ldots, \lambda'_i\}$, where λ'_i is the length of the ith column. A matrix $A \in \mathrm{GL}_3(\mathbb{C})$ acts via

$$(Af)(X) \stackrel{\text{def}}{=} f({}^tAX),$$

multiplying the argument X by the transposed matrix ${}^tA$.
Verify that a basis for V_λ is given by $\Delta^{I_1}_{[\lambda_1]}\Delta^{I_2}_{[\lambda_2]}\Delta^{I_3}_{[\lambda_3]}$ where I_1, I_2, I_3 run over the *column entries* of each semistandard tableau $T \in \mathrm{SSYT}_3(\lambda)$. Thus the character

$$\chi_\lambda(x_1, x_2, x_3) = \mathrm{trace}(\mathrm{diag}(x_1, x_2, x_3) \mid V_\lambda)$$

is equal to the Schur polynomial $s_\lambda(x_1, x_2, x_3)$.

c. For each V_λ, let U_λ be the subspace of vectors with character $x_1x_2x_3$, corresponding to *standard* Young tableaux T. Show that U_λ is a representation of the permutation matrix group $\mathfrak{S}_3 \subset \mathrm{GL}_3(\mathbb{C})$, and that this gives all the irreducible representations of $\mathfrak{S}_3$. (If necessary, look up the requisite facts about representations of finite groups.)

d. Consider the vector space V of matrices $X \in M_3(\mathbb{C})$ with zero trace: $\mathrm{tr}(X) = 0$. Define the *adjoint action* of $\mathrm{GL}_3(\mathbb{C})$ by conjugation:

$$\rho(A)X \;=\; AXA^{-1}.$$

Find the character $\chi_V)(x_1, x_2, x_3)$ of this representation and write it in terms of Schur polynomials, and hence identify the representation in terms of the V_λ.
Hint: Tensor by a power of the 1-dim determinant representation V_{111}.

e. Repeat (d) for the representation of $\mathrm{GL}_n(\mathbb{C})$ on traceless $n\times n$ matrices, which is the *adjoint representation* on the Lie algebra of $\mathrm{SL}_n(\mathbb{C})$. Write it in terms of Schur polyomials $s_\lambda(x_1, \ldots, x_n)$.