If you get significant help from a reference or a person, give explicit credit.

NOTES. Consider a poset $(\mathcal{P}, \leqslant)$ with minimal element $\hat{0}$. A rank function $\mathrm{rk} : \mathcal{P} \to \mathbb{N}$ is defined by $\mathrm{rk}(\hat{0}) = 0$ and $\mathrm{rk}(a) + 1 = \mathrm{rk}(b)$ for every covering $a \lessdot b$.

A *lattice* $\mathcal{L}$ is a poset having, for all $a, b \in \mathcal{L}$, a well-defined meet $a \wedge b = \inf\{a, b\}$ and join $a \vee b = \sup\{a, b\}$, and it is *distributive* if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, which is equivalent to $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$. Any finite distributive lattice $\mathcal{L}$ is isomorphic to the order ideals of its subposet $\mathcal{P}$ of join-irreducibles: that is, $\mathcal{L} \cong J(\mathcal{P}) \overset{\mathrm{def}}{=} \{I \subset \mathcal{P} \text{ with } (b \in I \ \&\ a \leqslant b) \Rightarrow a \in I\}$, ordered by inclusion.

A lattice is *semi-modular* if $\mathrm{rk}(a) + \mathrm{rk}(b) \geqslant \mathrm{rk}(a \wedge b) + \mathrm{rk}(a \vee b)$. An *atom* is a covering element $a \gtrdot \hat{0}$, and $\mathcal{L}$ is *atomic* if every element is a join of atoms (not necessarily uniquely). A lattice is *geometric* if it is semi-modular and atomic; this is equivalent to the combinatorial structure known as a *matroid*.

We define the *incidence algebra*:

$$I(\mathcal{P}) \;=\; \{\alpha : \mathrm{Int}(\mathcal{P}) \to \mathbb{C}\} \;\cong\; \textstyle\bigoplus_{a \leqslant b} \mathbb{C}[a, b], \qquad \alpha \cong \displaystyle\sum_{a<b} \alpha(a, b)\,[a, b],$$

all functions on the set $\mathrm{Int}(\mathcal{P})$ of intervals $[a, b] = \{c \text{ with } a \leqslant c \leqslant b\}$, with product:

$$(\alpha \cdot \beta)(a, b) = \sum_{c \in [a,b]} \alpha(a, c)\,\beta(c, b), \qquad [a, b] \cdot [c, d] = \begin{cases} [a, d] & \text{if } b = c \\ 0 & \text{otherwise.} \end{cases}$$

We can realize $I(\mathcal{P})$ as certain upper-triangular matrices: writing $\mathcal{P} = \{a_1, \ldots, a_n\}$ in non-decreasing order , $[a_i, a_j] \in I(\mathcal{P})$ is represented by the unit matrix $E_{ij} \in M_{n\times n}(\mathbb{C})$. The incidence algebra has identity element $\delta = \sum_{a\in\mathcal{P}} [a, a]$, zeta function $\zeta = \sum_{a\leqslant b} [a, b]$, and Mobius function $\mu = \zeta^{-1}$, so that $\zeta \cdot \mu = \delta$ is equivalent to the recursion: $\mu(a, a) = 1$ and $\sum_{c\in[a,b]} \mu(a, c) = 0$.

We consider several standard semi-infinite posets $\mathcal{P}_\infty$ which contain a minimal element $\hat{0}$ as well as standard elements $\hat{1}, \hat{2}, \ldots$, and which have a natural equivalence relation $[a, b] \sim [c, d]$ which splits $\mathrm{Int}(\mathcal{P})$ into equivalence classes $\bar{0}, \bar{1}, \bar{2}, \ldots$, where $\bar{n}$ is the equivalence class of $[\hat{0}, \hat{n}]$. We define the *reduced incidence algebra*:

$$R(\mathcal{P}_\infty) = \{\, \alpha \in I(\mathcal{P}_\infty) \text{ with } \alpha(a, b) = \alpha(c, d) \text{ for } [a, b] \sim [c, d] \,\} \;=\; \bigoplus_{n=0}^{\infty} \mathbb{C}\,\bar{n}.$$

We can write α as a kind of generating function for the sequence $a_n = \alpha(\hat{0}, \hat{n})$:

$$\alpha = a_0\bar{0} + a_1\bar{1} + a_2\bar{2} + a_3\bar{3} + \cdots .$$

$R(\mathcal{P}_\infty)$ contains the identity δ, the zeta function ζ, and the Möbius function $\mu = \zeta^{-1}$.

A *binomial poset* is a ranked poset $\mathcal{P}_\infty$ with a standard chain $\hat{0} \lessdot \hat{1} \lessdot \hat{2} \lessdot \cdots$, such that every interval $[a, b]$ with length $\ell(a, b) = \mathrm{rk}(b) - \mathrm{rk}(a) = n$ has $B(n)$ maximal chains. We let $[a, b] \sim [c, d]$ whenever $\ell(a, b) = \ell(c, d)$. The algebra $R(\mathcal{P})$ is isomorphic to the formal power series ring $\mathbb{C}[[x]]$, with $\bar{n}$ corresponding to $x^n/B(n)$, so that:

$$\alpha \cong a_0 + a_1 x + a_2 \tfrac{x^2}{B(2)} + a_3 \tfrac{x^3}{B(3)} + \cdots .$$

The posets $\mathcal{D}_n$ of divisors of n have the semi-infinite union $\mathcal{P} = \mathcal{D}_\infty = \{1, 2, 3, \ldots\}$ ordered by divisibility. This has standard elements $\hat{n} = n$ (which do not lie on a chain), and the equivalence of intervals is $[a, b] \sim [c, d]$ whenever $b/a = d/c$,[1] which induces equivalence classes $\bar{n} = \overline{[1, n]}$, making a basis of the reduced algebra $R(\mathcal{P}) = \bigoplus_{n\geqslant 1} \mathbb{C}\,\bar{n}$. We have $\bar{n}\bar{m} = \overline{nm}$, so $R(\mathcal{P})$ embeds in the ring of complex functions via $\bar{n} \cong n^{-s}$, where s is a complex variable, so that $\alpha \cong \sum_{n\geqslant 1} a_n/n^s$.

[1] This is stronger than rank equivalence $\ell(a, b) = \ell(c, d)$ and isomorphism equivalence $[a, b] \cong [c, d]$.

Problems

1. Direct product of posets: $\mathcal{P}\times\mathcal{Q}$, with $(p,q)\leqslant(p',q')$ whenever $p\leqslant p'$ and $q\leqslant q'$. Prove: Möbius function of the product is the product of Möbius functions of $\mathcal{P}$ and $\mathcal{Q}$:

$$\mu_{\mathcal{P}\times\mathcal{Q}}((p,q),(p',q')) \;=\; \mu_{\mathcal{P}}(p,p')\;\mu_{\mathcal{Q}}(q,q').$$

2. For the poset $\mathcal{P}=\mathcal{D}_{18}$, the 6-element poset of divisors of $18=2\cdot 3^2$ ordered by divisibility, work out the Möbius function $\mu(a,b)$ in several ways:

a. Write a 6×6 matrix Z corresponding to $\zeta(a,b)=1$ for all $a\leqslant b$, and invert by Gaussian elimination on a double matrix $[Z\,|\,I\,]$, row reducing to $[\,I\,|\,M]$ so $M=Z^{-1}$.

b. Write $Z=I+N$, for identity I and strictly upper-triangular N, nilpotent with $N^6=0$. By computer, expand the geometric series $M=(I+N)^{-1}=I-N+N^2-\cdots$.

c. For each $a\in\mathcal{P}$, draw a copy of the Hasse diagram (a 1×2 rectangle). Mark $\mu(a,a)=1$, then work upwards recursively using $\mu(a,b)=-\sum_{a\leqslant x<b}\mu(a,x)$.

d. Repeat (c) using the Weisner recurrence, within each $[a,b]$ choosing some $c>a$:

$$\mu(a,b)=-\sum_{x}\mu(a,x)\ \text{ summed over } x\in[a,b] \text{ with } c\nleqslant x<b.$$

Do only for the $[a,b]$ where Weisner uses fewer terms than the full recurrence. Also look up: when is the Weisner recurrence valid?

e. Apply the product formula of #1 above to $\mathcal{D}_{18}\cong[0,1]\times[0,2]$, the direct product of two chains. Match this with Mobius' original formula for $\mu(n)=\mu(1,n)$: namely $\mu(n)=(-1)^k$ if n is the product of k distinct primes, and $\mu(n)=0$ if n is divisible by a square number.

3. Recall the Euler phi-function $\phi(n)=\#\{i\in[n] \text{ with } \gcd(i,n)=1\}$.

a. Find a summation formula for $\sum_{d|n}\phi(d)$, and use Mobius inversion for $\mathcal{P}=\mathcal{D}_\infty$ to express $\phi(n)$ in terms of $\mu(n)=\mu(1,n)$.

b. Use the isomorphism of $R(\mathcal{D}_\infty)$ with the ring of Dirichlet series to find an expression for $f(s)=\sum_{n\geqslant 1}\phi(n)/n^s$ in terms of the Riemann zeta function $\zeta(s)=\sum_{n\geqslant 1}1/n^s$.

c. Use (b) to show that $\phi(n)$ grows sufficiently fast that $\sum_{n\geqslant 1}\phi(n)/n^2$ is divergent. Thus, there is no $\varepsilon>0$ such that $\phi(n)=\mathrm{o}(n^{1-\varepsilon})$. (For a challenge, be as analytically rigorous about this as possible. Perhaps learn about Perron's formula.)

4. Zaslavsky's Theorem (second part): Consider an affine hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ with characteristic polynomial $\chi_{\mathcal{A}}(x)=\sum_{V\in\mathcal{L}(\mathcal{A})}\mu(\hat{0},V)\,x^{\dim(V)}$. Then the hyperplane complement

$$\mathbb{R}^n\setminus\bigcup_{H\in\mathcal{A}}H$$

has $b(\mathcal{A})$ *bounded* regions, where:

$$b(\mathcal{A})=(-1)^n\chi_{\mathcal{A}}(1).$$

Prove this using an appropriate Deletion-Restriction recurrence among $b(\mathcal{A})$, $b(\mathcal{A}')$, and $b(\mathcal{A}'')$, where we choose $H_\circ\in\mathcal{A}$ and take $\mathcal{A}'=\mathcal{A}\setminus\{H_\circ\}$, and

$$\mathcal{A}''=\{H\cap H_\circ\subset H_\circ \text{ for } H\in\mathcal{A}'\},$$

an arrangement of hyperplanes inside the $(n-1)$-dimensional space H_0. Match this with the known Deletion-Restriction recurrence $\chi_{\mathcal{A}}(x)=\chi_{\mathcal{A}'}(x)-\chi_{\mathcal{A}''}(x)$.

5. For a finite field $F = \mathbb{F}_q$, recall the q-Boolean poset $\mathcal{B}_n(q)$ of linear subspaces $V \subset F^n$ ordered by inclusion. The semi-infinite union $\mathcal{B}_1(q) \subset \mathcal{B}_2(q) \subset \cdots$ via $F^1 \subset F^2 \subset \cdots$ is the Boolean poset $\mathcal{P}_\infty = \mathcal{B}_\infty(q)$, with standard elements $\hat{n} = F^n$. The reduced incidence algebra $R(\mathcal{P}_\infty)$ is isomorphic to $\mathbb{C}[[x]]$, with basis element $\bar{n}$ corresponding to $x^n/[n]_q^!$, the ring of Eulerian generating functions:

$$f(x) \;=\; \sum_{n \geqslant 0} a_n \frac{x^n}{[n]_q^!} \;=\; a_0 + a_1 x + a_2 \frac{x^2}{1+q} + a_3 \frac{x^3}{(1+q+q^2)(1+q)} + \cdots .$$

a. Explain why $\mathcal{P}_\infty$ is a binomial poset with

$$B(n) = [n]_q^! = \#\mathrm{Flag}(\mathbb{F}_q^n) = [n]_q [n-1]_q \cdots [2]_q [1]_q, \quad \text{where } [n]_q = \tfrac{q^n - 1}{q-1}.$$

b. Show the reciprocal of $\zeta = \sum_{n \geqslant 0} \frac{x^n}{[n]_q^!} \overset{\text{def}}{=} \exp_q(x)$ is the power series

$$\mu \;=\; \sum_{n \geqslant 0} (-1)^n q^{\binom{n}{2}} \frac{x^n}{[n]_q^!} \;\overset{\text{def}}{=}\; \mathrm{Exp}_q(-x),$$

and determine the Mobius function $\mu(U, V)$ for any $U \subset V$ in $\mathcal{B}_\infty(q)$.
Hint: Simplify $\zeta \cdot \mu$ with q-binomial thm $\prod_{i=1}^n (1 + q^{i-1}x) = \sum_{k=0}^n q^{\binom{k}{2}} \binom{n}{k}_q x^k$ (HW 5).
Note: We determined μ in class using the Weisner recurrence: $\mu_n = (-q^{n-1})\mu_{n-1}$.

c. For an n-dimensional $V \in \mathcal{B}_n(q)$ and a fixed F^N, let

$$\alpha(V) \;=\; \#\{\text{linear } f : F^N \to V\} \;=\; q^{Nn},$$

$$\beta(V) \;=\; \#\{\text{surjective linear } f : F^N \twoheadrightarrow V\}$$

Show by linear algebra that $\alpha(V) = \sum_{W \subset V} \beta(W)$, and solve for $\beta(V)$ by Mobius inversion, a summation formula for the number of surjective $f : F^N \twoheadrightarrow F^n$.

Is $\beta(n)$ a good q-analog of $\mathrm{surj}(N, n) = \#\{\text{ surjective } f : [N] \twoheadrightarrow [n]\,\}$?

d. Determine the number of injective linear mappings $f : F^n \hookrightarrow F^N$ directly by a simple product formula.

e. Show by linear algebra that the number of surjective linear mappings $f : F^N \twoheadrightarrow F^n$ is equal to the number of injective linear mappings $f : F^n \hookrightarrow F^N$.

f. Use the identity between the product in (d) and the sum in (c) to give another proof of the q-binomial theorm.