Math 880 **Homework 5** **Due Mon 10/29**

If you get significant help from a reference or a person, give explicit credit.

NOTES

The Grassmannian $\mathrm{Gr}(k, F^n)$ is the parameter space whose points correspond to k-dimensional subspaces V in the n-dimensional vector space F^n over a given field F. We specify a subspace $V = \mathrm{Span}_F(v_1, \ldots, v_k)$ by a $k \times n$ matrix of row vectors, with change-of-basis symmetry group $\mathrm{GL}_k(F)$. This matrix can be normalized by making a given $k \times k$ submatrix into the identity, in columns $I = \{i_1 < \cdots < i_k\} \subset [n]$, provided the determinant of this submatrix is nonzero:

$$V \;=\; \mathrm{GL}_k \circlearrowright \begin{bmatrix} \text{———} \; v_1 \; \text{———} \\ \text{———} \; v_2 \; \text{———} \\ \vdots \\ \text{———} \; v_k \; \text{———} \end{bmatrix} \;=\; \begin{bmatrix} * \; * \cdots 1 \cdots 0 \cdots 0 \cdots * \; * \\ * \; * \cdots 0 \cdots 1 \cdots 0 \cdots * \; * \\ \vdots \\ * \; * \cdots 0 \cdots 0 \cdots 1 \cdots * \; * \end{bmatrix}$$

The $*$'s denote $k(n-k)$ free parameters in F defining a coordinate chart U_I of the Grassmannian, an F-manifold (nonsingular algebraic variety): $\mathrm{Gr}(k, F^n) = \bigcup_I U_I$.

We define the Schubert cell decomposition $\mathrm{Gr}(k, F^n) = \coprod_I X_I$ by letting X_I consist of those $V \in U_I$ which have no $*$'s to the right of any 1 (row-echelon form). Thus, X_I is an affine space of dimension $\mathrm{wt}(I) = \sum_{j=1}^{k}(i_j - j)$. We can define X_I geometrically in terms of the standard basis $\{e_1, \ldots, e_n\}$ of F^n and the standard coordinate subspaces $E_r = \mathrm{Span}(e_1, \ldots, e_r)$:

$$V \in X_I \quad\Longleftrightarrow\quad \dim(V \cap E_r) = \#(I \cap [r]) \text{ for } r = 1, \ldots, n.$$

That is, $I = [k]$ forces $V = E_k$, and larger I makes $V \in X_I$ stick out further from the standard subspaces, until $I = \{n-k+1, \ldots, n\}$ corresponds to generic V's in the open set $X_I = U_I$. The topological closure $\overline{X}_I$ is defined by the closed conditions: $\dim(V \cap E_r) \geqslant \#(I \cap [r])$ for $r = 1, \ldots, n$. We keep track of how cells fit together using Bruhat degeneration order: define $I \leqslant J$ to mean $X_I \subset \overline{X}_J$, or equivalently $\overline{X}_I \subset \overline{X}_J$.

EXAMPLE: For $\mathrm{Gr}(2, F^4)$, we have:

$$U_{34} \;=\; X_{34} \;=\; \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix} \;=\; \{V \mid V \cap E_2 = 0\,, \dim(V \cap E_3) = 1\},$$

$$U_{14} \;=\; \begin{bmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{bmatrix}, \qquad X_{14} \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix} \;=\; \{V \mid E_1 \subset V \not\subset E_3\}.$$

Here are the cells with defining conditions, vertical dashes indicating Bruhat order:

$$\begin{array}{ccc} & X_{34} = (V\cap E_2 = 0) & \\ & | & \\ & X_{24} = (\dim(V\cap E_2) \geqslant 1, V\not\subset E_3) & \\ & / \qquad \backslash & \\ X_{23} = (E_1\not\subset V\subset E_3) & & X_{14} = (E_1\subset V\not\subset E_3) \\ & \backslash \qquad / & \\ & X_{13} = (E_1\subset V\subset E_3, V\neq E_2) & \\ & | & \\ & X_{12} = (V=E_2) & \end{array}$$

The closure of each cell is defined by keeping the closed conditions on V such as $\dim(V\cap E_2) \geqslant 1$, and ignoring the open conditions such as $V\not\subset E_3$. The Bruhat order relations $\overline{X}_I \subset \overline{X}_J$ are thus evident from these closed conditions. To verify in coordinates that $\{1,3\} \leqslant \{1,4\}$, we show each plane $V_\circ \in X_{13}$ is approached by planes in X_{14}: we give a continuous $\mathcal{V} : F \to \mathrm{Gr}(2, F^4)$ with $\mathcal{V}(t) \in X_{14}$ for $t \neq 0$, and $\mathcal{V}(0) = V_\circ$.

$$V_\circ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 \end{bmatrix}, \qquad \mathcal{V}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a/t & 1/t & 1 \end{bmatrix} \text{ for } t \neq 0.$$

Next we consider the flag manifold $\mathrm{Flag}(F^n)$, the parameter space of flags

$$V_\bullet = (0 \subset V_1 \subset \cdots \subset V_{n-1} \subset F^n), \quad \dim(V_k) = k.$$

We specify $V_\bullet$ by a basis $\{v_1, \ldots, v_n\}$ of F^n, with $V_k = \mathrm{Span}(v_1, \ldots, v_k)$; the basis forms an $n\times n$ matrix of row vectors. The change-of-basis symmetry group B of $V_\bullet$ consists of all *lower*-triangular matrices (with non-zero diagonal entries) in $\mathrm{GL}_n(F)$, since we can add a multiple of v_i to any later basis vector v_j with $j > i$, leaving each V_k invariant. We get a Schubert cell decomposition indexed by permutations $w \in \mathfrak{S}_n$: $\mathrm{Fl}(F^n) = \coprod_w X_w$, where X_w consists of $V_\bullet$ whose B-reduced form is a permutation matrix w with 1 in positions $(i, w(i))$, and free parameters $*$ in the positions (i, j) of the Röthe diagram:

$$D(w) = \{(i,j) \mid j < w(i),\ i < w^{-1}(j)\}.$$

Thus $\dim(X_w)$ is an affine space of dimension $\#D(w)$.

PROBLEMS

1. For a permutation $w \in \mathfrak{S}_n$, define the inversion number

$$\mathrm{inv}(w) = \#\{(i,j) \mid i < j \text{ and } w(i) > w(j)\}.$$

Prove that $\#D(w) = \mathrm{inv}(w)$. *Hint:* Bijection.

Note: The inversion number is also equal to the length $\ell(w)$, the minimum number ℓ of adjacent transpositions $s_i = (i, i+1)$ in a decomposition $w = s_{i_1} \cdots s_{i_\ell}$. Also $u \leqslant w$ whenever u is equal to a subword of this decomposition, for example $s_1 s_3 \leqslant s_1 s_2 s_3 s_1$.

2a. Compute the Gaussian binomial coefficient $\left[{6 \atop 3}\right]_q = \#\mathrm{Gr}(3, \mathbb{F}_q^6)$ as the number of $3{\times}6$ V-basis matrices divided by the number of $3{\times}3$ change-of-basis matrices; also as a quotient of q-integers $[n]_q = \frac{q^n - 1}{q-1}$; and finally as a polynomial by dividing through (with computer).

b. There are $\binom{6}{3} = 20$ sets $I = \{i_1, i_2, i_3\} \subset [6]$ indexing the Schubert cells, in bijection with partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0)$ with $\lambda_1 \leqslant 6{-}3$: the Young diagrams fit in a $3{\times}3$ rectangle. List all I's and λ's, along with the size measure $q^{\mathrm{wt}(I)} = q^{|\lambda|} = \#X_I$. Here $\mathrm{wt}(I) = |\lambda| = \dim(X_I)$. Compare with part (a).

3a. Verify the q-Binomial Theorem:

$$\prod_{i=1}^{n} (1 + q^i x) \;=\; \sum_{k=0}^{n} q^{\binom{k+1}{2}} \binom{n}{k}_q x^k$$

for the special case $n = 3$. Multiply out by hand!

b. Prove the q-Binomial Theorem for all n by writing the lefthand side as a bivariate generating function for the class of all subsets $I \subset [n]$, then using our expansion illustrated in Prob. 1, $\binom{n}{k}_q = \sum_I q^{\mathrm{wt}(I)}$, where the sum is over all $I = \{i_1 < \cdots < i_k\}$.

4. Find q-analogs of the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ using two analogs of deletion.

a. Consider the mapping $\mathrm{Gr}(k, F^n) \to \mathrm{Gr}(k, F^{n-1}) \coprod \mathrm{Gr}(k{-}1, F^{n-1})$ which takes V to $V' = V \cap E_{n-1}$, intersecting with the coordinate subspace E_{n-1}. Make this a bijection by keeping track of lost information needed to reconstruct the row-echelon basis of V from V'. Deduce a recurrence for $\binom{n}{k}_q$.

b. Let $P : F^n \to F^{n-1}$ be the projection map along e_1, so that $P(e_i) = e_i$ for $i = 2, \ldots, n$. Consider the mapping $\mathrm{Gr}(k, F^n) \to \mathrm{Gr}(k, F^{n-1}) \coprod \mathrm{Gr}(k{-}1, F^{n-1})$ of V to $V' = P(V)$. Again make this a bijection, and find a different recurrence for for $\binom{n}{k}_q$.

5a. For each Schubert cell $X_w \subset \mathrm{Flag}(F^3)$ corresponding to $w \in S_3$, write the B-reduced matrix form for $V_\bullet$ corresponding to the Röthe diagram $D(w)$. For $F = \mathbb{F}_q$, explicitly verify:

$$\#\mathrm{Flag}(\mathbb{F}_q^3) \;=\; \frac{\#\mathrm{GL}_3(\mathbb{F}_q)}{\#B} \;=\; [3]_q[2]_q[1]_q \;=\; \sum_{w \in S_3} q^{\mathrm{inv}(w)}.$$

b. For each Schubert cell closure $\overline{X}_w$ above, describe its flags $V_\bullet = (V_1 \subset V_2)$ in terms of their relation to the standard flag $E_1 \subset E_2$. For example, the minimal cell closure is: $\overline{X}_{123} = X_{123} = \{E_\bullet\} = \{V_\bullet \mid V_1 = E_1\,,\ V_2 = E_2\}$. By examining the implications among the defining conditions for the $\overline{X}_w$'s, arrange the w's according their Bruhat degeneration order, defined by $\overline{X}_w \subset \overline{X}_u$.

c. The Bruhat order covering relations $u \lessdot w$ correspond to minimal containments $\overline{X}_u \subset \overline{X}_w$, having $\dim X_u = \dim X_w - 1$. For each minimal containment in $\mathrm{Flag}(F^3)$ and any $(V_\bullet) \in X_u$, we can show the relation by giving a continuous family $\mathcal{V} : F \to \mathrm{Fl}(F^3)$ with $\mathcal{V}(t) \in X_w$ for $t \neq 0$ and $\mathcal{V}(0) = V_\bullet$.

Example: I claim $u = 132 \lessdot w = 231$. By row-echelon reduction for $t \neq 0$:

$$\mathcal{V}(t) = B\cdot\begin{bmatrix} 1 & -t & 0 \\ 0 & a & 1 \\ 0 & 1 & 0 \end{bmatrix} = B\cdot\begin{bmatrix} -1/t & 1 & 0 \\ a/t & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in X_{231}, \quad \mathcal{V}(0) = B\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 1 \\ 0 & 1 & 0 \end{bmatrix} \in X_{132}.$$

Problem: Similarly prove the covering $u = 213 < w = 312$.

d. A combinatorial construction for the coverings $u \lessdot w$ is based on their permutation matrices: for a pair of 1's making an inversion in w ($i < j$ and $w(i) > j$), move them to form a non-inversion pair in $u = (i, j) \cdot w$.

Problem: Find an additional condition on the matrices so that $u \lessdot w$ is a covering, with $\mathrm{inv}(u) = \mathrm{inv}(w) - 1$. For example, why does $213 = (2,3) \cdot 312$ give a covering, while $123 = (1,3) \cdot 321$ does not?