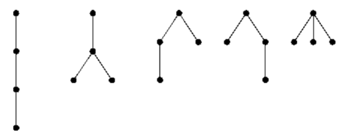
Math 880 **Homework 3** **Fall 2023**

Feel free to discuss homework problems with other students, and to learn from references, but please do not look up specific answers. Write out solutions in your own words, and give explicit credit for any significant help.

1. *Tree Structures.* A *plane tree* is a combinatorial graph with k unlabeled vertices: an ancestor (root vertex) at the top, and below each vertex is a list of children in age order. Let $\mathcal{B}_k$ be the class of such trees, B_k the counting number. E.g. $B_4 = 5$:



The difference between the 3rd and 4th trees is whether the ancestor's older or younger child has a child. Set $\mathcal{B}_0 = \{\}$. Deleting the root gives recursively:

$$\mathcal{B} \;\cong\; \{\bullet\} \times \textsc{Seq}(\mathcal{B}),$$

where $\{\bullet\}$ denotes the class with a single element of size 1, and

$$\textsc{Seq}(\mathcal{B}) \;=\; \{\varnothing\} \;\sqcup\; \mathcal{B} \;\sqcup\; \mathcal{B}\times\mathcal{B} \;\sqcup\; \cdots$$

denotes all lists of elements in $\mathcal{B}$. By the Graded Product Principle:

$$B(x) \;=\; x(1 + B(x) + B(x)^2 + \cdots) \;=\; \frac{x}{1 - B(x)}.$$

Solving gives $B(x) = \frac{1}{2}(1 - \sqrt{1-4x})$; comparing this to the Catalan generating function implies $B_k = C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1}$.

The Catalan formula can be obtained easily by Lagrange inversion: if a power series $f(x)$ has inverse function $g(x)$, so that $f(g(x)) = x$, then the x^k coefficient of $g(x) = f^{-1}(x)$ is given by: $\frac{1}{k}[x^{-1}]\frac{1}{f(x)^k}$. Here $B(x)(1-B(x)) = x$, so $f(x) = x(1-x)$.

Problem: Use the above techniques to count the following types of rooted trees:

a. Binary plane trees: k unlabeled vertices, no children or two ordered children.

b. Labeled plane trees: k *labeled* vertices, any number of ordered children.

c. Increasing trees: k labeled vertices, any number of *unordered* children, and each child must have a larger label than its parent (so the root must be 1). For example, $k = 2$ gives two trees: $1{-}2{-}3$ and $1{-}\{2,3\}$. *Hint:* Use the directed labeled product $\mathcal{A}^{\min} * \mathcal{B}$. *Extra Credit:* Bijectively prove the simple formula that results.

d. Ternary trees: k unlabeled vertices with the root at top, and each vertex having zero or three ordered children. Hint: Any ternary tree has $k = 3j + 1$ vertices.

2. *Burnside Lemma:* For group G acting on set $\mathcal{F}$, the orbit set $\overline{\mathcal{F}} = \mathcal{F}/G = \{Gf \mid f \in \mathcal{F}\}$ is counted by the average number of fixed points $\mathcal{F}^g = \{f \in \mathcal{F} \,|\, gf = f\}$:

$$\#\bar{\mathcal{F}} \;=\; \frac{1}{\#G} \sum_{g\in G} \#\mathcal{F}^g\,.$$

If $\mathcal{F}$ is graded and G acts on each $\mathcal{F}_k$, we get the generating function formula:

$$\bar{F}(x) \;=\; \frac{1}{\#G} \sum_{g\in G} F^g(x).$$

a. Define the *Euler phi-function* $\phi(k) = \#\{j \le k \mid \gcd(j,k) = 1\}$, the number of integers in $[k]$ which are relatively prime to k. For example, $\phi(p) = p{-}1$ for p prime. Use Inclusion-Exclusion to prove that if k has prime factorization $k = p_1^{\ell_1} p_2^{\ell_2} \cdots$, then $\phi(k) = \prod_i p_i^{\ell_i-1}(p_i{-}1)$.

b. For a group of permutations $G \subset S_k$, define the *cycle index polynomial*:

$$Z_G(z_1,\ldots,z_k) \;=\; \sum\nolimits_{g\in G} z_1^{c_1(g)} \cdots z_k^{c_k(g)},$$

where $c_i(g) = \#$ i-cycles of g. For example, $G = S_3$ has $Z_G = z_1^3 + 3z_1z_2 + 2z_3$.

Determine the cycle index $Z_G(z_1,\ldots,z_k)$ for $G = C_k$, the cyclic group generated by $(12\cdots k) \in S_k$. The answer will involve the phi-function.

c. Let $\mathcal{F} = \{f : [k] \to \mathcal{A}\} \cong \mathcal{A}^k$ be functions from $[k]$ to graded class $\mathcal{A}$, with $|f| = \sum_{i=1}^k |f(i)|$, and let $G \subset \mathfrak{S}_k$ permute $[k]$, inducing an action on $\mathcal{F}$. The fixed-point set $\mathcal{F}^g$ consists of $f : \{\text{cycles of } g\} \to \mathcal{A}$, functions constant on each cycle of g. Use graded counting principles to compute the generating functions of $\mathcal{F}^g$ and $\overline{\mathcal{F}}$:

$$F^g(x) = A(x)^{c_1(g)} A(x^2)^{c_2(g)} \cdots A(x^k)^{c_k(g)},$$
$$\bar{F}(x) = \tfrac{1}{\#G} Z_G(A(x), A(x^2), \ldots, A(x^k)).$$

d. For a graded unlabeled class $\mathcal{A}$, consider the class $\overline{\mathcal{F}} = \textsc{Cyc}_k\,\mathcal{A}$ consisting of all cyclic arrangements of k elements of $\mathcal{A}$, up to rotation symmetry. Use the previous problems to find the generating function $\bar{F}(x)$ in terms of $A(x)$.

3. Let $\tilde{G}(x)$ be the exponential generating function of graphs on on k vertices labeled $\{1,\ldots,k\}$, and $\tilde{C}(x)$ the corresponding function for connected graphs. A graph is a set of connected subgraphs; the Exponential Formula implies $\tilde{G}(x) = \exp\,\tilde{C}(x)$, so:

$$\tilde{C}(x) \;=\; \sum_{k\ge1} c_k \frac{x^k}{k!} \;=\; \log\,\tilde{G}(x) \;=\; \log\left(1 + \sum_{k\ge1} 2^{\binom{k}{2}} \frac{x^k}{k!}\right),$$

where the outside function of the composition is the series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$. (Use small c_k to avoid confusion with Catalan numbers.)

a. Find a way to evaluate this series in Wolfram Alpha or Mathematica, at least far enough to compute c_k for $k = 4$. Verify the result by directly enumerating connected graphs on 4 vertices. Hint: First find the unlabeled graphs.

b. Use the *Dlog Method* to find a recurrence for c_k. That is, given $e^{\tilde{C}(x)} = \tilde{G}(x)$, perform logarithmic differentiation $\frac{d}{dx}\log$ to both sides, clear denominators, and equate coefficients on the two sides. Finally, solve for c_{k+1}.