Math 880 **Homework 2** **Fall 2023**

Feel free to discuss homework problems with other students, and to learn from references, but please do not look up specific answers. Write out solutions in your own words, and give explicit credit for any significant help. LaTeX is encouraged, but not required.

1. Recall a partition of k is a set of positive integer parts $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots\}$ adding up to $\lambda_1 + \lambda_2 + \cdots = k$, and $p(k)$ is the number of partitions of k. Transforming in terms of multiplicities, we write $\lambda = 1^{m_1}2^{m_2}\cdots$, where m_i is the number of parts $\lambda_j = i$; then the Graded Product Principle leads to the generating function:

$$P(x) \;=\; \sum_{k\geq 0} p(k)x^k \;=\; \prod_{i\geq 1}\frac{1}{1-x^i}.$$

Now let $q_{\leq 2}(k)$ be the number of partitions of k with no part repeated three or more times: for example $q_{\leq 2}(4) = 4$ counts 4, 3+1, 2+2, 2+1+1, but not 1+1+1+1 which contains 1 repeated four times. Also, let $p_{\nmid 3}(k)$ be the number of partitions of k with no part divisible by 3: for example $p_{\nmid 3}(4) = 4$ counts 4, 2+2, 2+1+1, 1+1+1+1, but not 3+1, with part 3.

Problem: Prove that $q_{\leq 2}(k) = p_{\nmid 3}(k)$ for all k by showing the equality of generating functions $Q_{\leq 2}(x) = P_{\nmid 3}(x)$. Hint: $(1+x+x^2)(1-x) = 1-x^3$.

2. An easy computational exercise. Using Euler's Pentagonal Number Theorem $\frac{1}{P(x)} = \prod_{i\geq 1}(1-x^i) = 1 + \sum_{\ell\geq 1}(-1)^\ell(x^{\frac{1}{2}\ell(3\ell-1)} + x^{\frac{1}{2}\ell(3\ell+1)})$, we obtain the following formula for the partition numbers $p(k)$:

$$\left(\sum_{k\geq 0} p(k)x^k\right)\left(1 + \sum_{\ell\geq 1}(-1)^\ell(x^{\frac{1}{2}\ell(3\ell-1)} + x^{\frac{1}{2}\ell(3\ell+1)})\right) \;=\; 1.$$

Write the resulting recurrence for $p(k)$, and use it to compute $p(0), p(1), \ldots, p(6)$. Explicitly write the 11 partitions for $p(6)$ by parts as $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with $\sum_i \lambda_i = 6$, and by multiplicities as $1^{m_1}2^{m_2}\cdots$.

Recall constructions for labeled $\tilde{\mathcal{C}}$ with exponential generating function $\tilde{C}(x) = \sum_{k\geq 0} C_k \frac{x^k}{k!}$.

$\tilde{\mathcal{C}} = \tilde{\mathcal{A}} * \tilde{\mathcal{B}}$,	$\tilde{C}(x) = \tilde{A}(x)\tilde{B}(x)$	$\tilde{\mathcal{C}} = \tilde{\mathcal{A}}^{\min} * \tilde{\mathcal{B}}$,	$\tilde{C}(x) = \int \tilde{A}'(x)\tilde{B}(x)\,dx$
$\tilde{\mathcal{C}} = \widetilde{\mathrm{SEQ}}_n\tilde{\mathcal{A}}$,	$\tilde{C}(x) = \tilde{A}(x)^n$	$\tilde{\mathcal{C}} = \widetilde{\mathrm{SEQ}}\,\tilde{\mathcal{A}}$,	$\tilde{C}(x) = 1/(1-\tilde{A}(x))$
$\tilde{\mathcal{C}} = \widetilde{\mathrm{SET}}_n\tilde{\mathcal{A}}$,	$\tilde{C}(x) = \tilde{A}(x)^n/n!$	$\tilde{\mathcal{C}} = \widetilde{\mathrm{SET}}\,\tilde{\mathcal{A}}$,	$\tilde{C}(x) = \exp\tilde{A}(x)$
$\tilde{\mathcal{C}} = \widetilde{\mathrm{CYC}}_n\tilde{\mathcal{A}}$,	$\tilde{C}(x) = \tilde{A}(x)^n/n$	$\tilde{\mathcal{C}} = \widetilde{\mathrm{CYC}}\,\tilde{\mathcal{A}}$,	$\tilde{C}(x) = -\log(1-\tilde{A}(x))$

3. Recall the Euler number E_k which counts the down-up permutations $\pi \in \mathfrak{S}_k$ with $\pi(1) > \pi(2) < \pi(3) > \cdots$. For example, $E_3 = 2$ counts the permutations $\pi = 213, 312$. We split these into the permutations $\tilde{\mathcal{J}}$ with odd length $k = 2\ell+1$, so that $J_{2\ell+1} = E_{2\ell+1}$ and $J_{2\ell} = 0$; and also the permutations $\tilde{\mathcal{K}}$ with even length $k = 2\ell$, so that $K_{2\ell} = E_{2\ell}$ and $K_{2\ell+1} = 0$.

Deleting $\pi(j) = 1$ from an odd permutation produced the recurrence:

$$\tilde{\mathcal{J}} \;\cong\; \tilde{\mathcal{J}} \star [1]^{\min} \star \tilde{\mathcal{J}} \;+\; [1],$$

which can be solved via generating functions to give $\tilde{J}(x) = \tan(x)$.

a. Use a similar method to determine $\tilde{K}(x)$.

b. Check that the smallest complex singularities of $\tilde{E}(x) = \tilde{J}(x) + \tilde{K}(x)$ are $\alpha = \frac{\pi}{2}, \beta = -\frac{\pi}{2}$. Find the residues A, B such that:

$$\tilde{E}(x) \;=\; \frac{A}{1 - x/\alpha} + \frac{B}{1 - x/\beta} + f(x),$$

where $f(x)$ is analytic (non-singular) on $|x| \leq \frac{3}{2}\pi$. Hints: In the complex plane $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$. The residue of $\tilde{E}(x)$ at $x = \alpha$ is computed by $A = \lim_{x \to \alpha} \tilde{E}(x)(1 - x/\alpha)$.

By Flajolet-Sedgewick Thm IV.10 p. 258, this implies the asymptotic expansion

$$\frac{E_k}{k!} \sim A\alpha^{-k} + B\beta^{-k}.$$

Use Stirling's approximation $k! \sim \sqrt{2\pi k}\,(k/e)^k$ to get a simple asymptotic for E_k. Look up E_{99} and E_{100} (in a standard Taylor series) to see how accurate this is in percentage terms.

4. The class of *derangements* $\tilde{\mathcal{D}}_k$ comprises all permutations $\pi \in \mathfrak{S}_k$ with $\pi(i) \neq i$ for all i. We saw that $\tilde{\mathcal{D}} \cong \tilde{\text{S}\text{ET}}(\tilde{\text{C}\text{YC}}_{\geq 2}[1])$, which gives:

$$\tilde{D}(x) = \exp\left(\log \tfrac{1}{1-x} - x\right) = \frac{e^{-x}}{1-x}, \qquad D_k = k!(1 - \tfrac{1}{1!} + \tfrac{1}{2!} - \cdots + (-1)^k \tfrac{1}{k!}).$$

Problem: Prove the above formula for D_k without generating functions, instead using the Principle of Inclusion-Exclusion. For fixed k, start by finding a bigger set $\mathcal{A}$ containing $\tilde{\mathcal{D}}_k$ as well as some bad sets $\mathcal{B}_1, \ldots, \mathcal{B}_k$ such that $\tilde{\mathcal{D}}_k = \mathcal{A} \setminus (\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k)$.

5. A labeled signed graded class is split into positive and negative elements $\tilde{\mathcal{A}}_k^{\pm} = \tilde{\mathcal{A}}_k^{+} \cup \tilde{\mathcal{A}}_k^{-}$, and has the signed generating function:

$$\tilde{A}^{\pm}(x) \;=\; \sum_{k \geq 0} (A_k^{+} - A_k^{-}) \frac{x^k}{k!}.$$

a. The *Labeled Graded Product Principle* states that a labeled Cartesian product $\tilde{\mathcal{A}} \star \tilde{\mathcal{B}}$ has generating function $\tilde{A}(x)\tilde{B}(x)$.

Problem: Prove that this also holds for the Cartesian product of *signed* labeled graded classes (with an appropriate definition of sgn).

b. The *Involution Priniciple* assumes a bijection $I : \tilde{\mathcal{A}}_k^{\pm} \to \tilde{\mathcal{A}}_k^{\pm}$ with $I^{-1} = I$ which reverses sign, $\text{sgn}(I(a)) = -\text{sgn}(a)$, except on the fixed-point set $\tilde{\mathcal{F}}_k^{\pm} = \{a \in \tilde{\mathcal{A}}_k^{\pm} \mid I(a) = a\}$. In the non-fixed part of $\tilde{\mathcal{A}}_k^{\pm}$, the involution pairs positive and negative elements, leaving only the fixed points unpaired; thus the we get the equality of signed generating functions:

$$\tilde{F}^{\pm}(x) = \tilde{A}^{\pm}(x).$$

Problem: Prove the equation:

$$\tilde{D}(x) \;=\; \frac{1}{1-x} \cdot e^{-x}$$

by interpreting the right side as the product of signed exponential generating functions for two simple signed labeled graded classes. Then find an involution whose fixed points (all positive) are in bijection with derangements:

$$\tilde{\mathcal{F}}^{\pm} \;=\; \tilde{\mathcal{F}}^{+} \;\cong\; \tilde{\mathcal{D}}_k.$$