**Math 880** **Homework 1** **Fall 2023**

Feel free to discuss homework problems with other students, and to learn from references, but please do not look up specific answers. Write out solutions in your own words, and give explicit credit for any significant help. LaTeX is encouraged, but not required.

The *Method of Generating Functions* consists of steps linking various more-or-less equivalent forms of combinatorial and analytic data:

(i) Graded class $\mathcal{A} = \coprod_{k\geq 0} \mathcal{A}_k$ of combinatorially defined objects $a \in \mathcal{A}$ with size $|a| = k$

(ii) Recurrence constructing $\mathcal{A}_k$ from smaller classes; recursive formula for $A_k = \#\mathcal{A}_k$

(iii) Explicit formula for the generating function $A(x) = \sum_{k\geq 0} A_k x^k = \sum_{a\in\mathcal{A}} x^{|a|}$

(iv) Explicit formula for $A_k$

(v) Asymptotic approximation $A_k \sim \alpha(k)$ for simple $\alpha(k)$, with $A_k/\alpha(k) \to 1$ as $k \to \infty$.

An enumeration problem usually starts with the first two forms of data, and the main goal is to derive the last two forms. The generating function is the central unifying object.

**1.** Define a *multiset* to be like a set, but with repeat elements allowed, such as $M = \{1, 3, 3, 5\}$. The number $\left(\!\binom{n}{k}\!\right)$, $n$ multi-choose $k$, counts all multisets of $k$ elements chosen from $[n] = \{1, 2, \ldots, n\}$. EXAMPLE: $\left(\!\binom{2}{3}\!\right) = 4$ counts $M = \{1, 1, 1\}, \{1, 1, 2\}, \{1, 2, 2\}, \{2, 2, 2\}$.

Analyze $\left(\!\binom{n}{k}\!\right)$ similarly to $\binom{n}{k}$, using the class $\mathcal{A}^{(n)} = \coprod_{k\geq 0} \mathcal{A}_k^{(n)}$ of all multisets from $[n]$.

(a) (i) $\Longrightarrow$ (iii). From the combinatorial definition of $\mathcal{A}^{(n)}$, find a multiset analog of the Bit-string Transformation of sets, which allowed us to apply the Product Principle to get $\sum_{k\geq 0} \binom{n}{k} x^k = (1+x)^n$. Similarly, derive a simple formula for $A^{(n)}(x) = \sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) x^k$.

(b) (i) $\Longrightarrow$ (ii). Use Deletion Transform to find a Pascal's Triangle type recurrence for $\left(\!\binom{n}{k}\!\right)$.

(c) (iii) $\Longrightarrow$ (ii). Derive the recurrence directly from the generating function by manipulating the denominator of $A^{(n)}(x)$.

(d) (iii) $\Longrightarrow$ (iv). Use the coefficient formula for Taylor series to get a simple formula for $\left(\!\binom{n}{k}\!\right)$, similar to $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$. Using this, find a binomial coefficient which is equal to the multi-choose number: $\left(\!\binom{n}{k}\!\right) = \binom{m}{k}$ for appropriate $m$.

(e) (i) $\Longrightarrow$ (iv). Give a bijection which proves the above identity $\left(\!\binom{n}{k}\!\right) = \binom{m}{k}$, transforming $k$-element multi-sets from $[n]$ into $k$-element sets from $[m]$.

(f) (iv) $\Longrightarrow$ (v). Show $\left(\!\binom{n}{k}\!\right) \sim \frac{k^{n-1}}{(n-1)!}$ for $k \to \infty$ and fixed $n$.

**2.** Recall the combinatorial class of surjections, doubly graded by $k \geq n$:

$$\mathcal{S}_k^n = \{\text{surjective functions } f : [k] \twoheadrightarrow [n]\},$$

and its counting number $\operatorname{surj}(k,n) = \#\mathcal{S}_k^n$. The two indexes $k, n$ produce two types of generating functions:

$$S^{(n)}(x) = \sum_{k\geq 0} \operatorname{surj}(k,n)\, x^k \qquad \text{and} \qquad S_{(k)}(y) = \sum_{n\geq 0} \operatorname{surj}(k,n)\, y^n.$$

(a) (i) $\Longrightarrow$ (ii). Use a Deletion Transform to show a Pascal-type recurrence for $\operatorname{surj}(k,n)$.

(b) (ii) $\Longrightarrow$ (iii). Use the recurrence to write $S^{(n)}(x)$ in terms of $S^{(n-1)}(x)$, and solve this to get a simple formula for $S^{(n)}(x)$.

(c) (iii) $\Longrightarrow$ (iv). Find the partial fraction expansion of $S^{(n)}(x)$, and a formula for $\operatorname{surj}(k,n)$. Compare to our formula from class using the Principle of Inclusion-Exclusion.