Unlabeled constructions. A *graded class* $\mathcal{A} = \coprod_{k\geq 0} \mathcal{A}_k$ is a set of combinatorial objects with size function $|a| = k \in \mathbb{N}$, and $\mathcal{A}_k = \{a \in \mathcal{A} \text{ with } |a| = k\}$. It has the counting sequence $A_k = \#\mathcal{A}_k$ and the ordinary generating function $A(x) = \sum_{a\in\mathcal{A}} x^{|a|} = \sum_{k\geq 0} A_k x^k$. The most important construction to combine classes is the Cartesian product $\mathcal{C} = \mathcal{A}\times\mathcal{B}$ graded by $|(a,b)| \overset{\text{def}}{=} |a| + |b|$, and having generating function $C(x) = A(x)B(x)$.

We also have the power construction, provided $\mathcal{A}_0 = \{\}$:

$$\mathcal{C} = \mathcal{B}^{\mathcal{A}} = \left\{\text{functions } f : \mathcal{A} \to \mathcal{B} \text{ with } |f| \overset{\text{def}}{=} \textstyle\sum_{a\in A} |a|\,|f(a)| < \infty\right\}, \quad C(x) = \prod_{i\geq 1} B(x^i)^{A_i}.$$

These result in the following standard constructions:

$$\begin{array}{ll} \mathcal{C} = \mathcal{A}\sqcup\mathcal{B},\ \ C(x) = A(x) + B(x) & \mathcal{C} = \mathcal{A}\times\mathcal{B},\ \ C(x) = A(x)B(x) \\ \mathcal{C} = \textsc{Seq}_n\mathcal{A},\ \ C(x) = A(x)^n & \mathcal{C} = \textsc{Seq}\,\mathcal{A},\ \ C(x) = 1/(1-A(x)) \\ \mathcal{C} = \textsc{Set}\,\mathcal{A},\ \ C(x) = \prod_{i\geq 1}(1+x^i)^{A_i} & \mathcal{C} = \textsc{MSet}\,\mathcal{A},\ \ C(x) = \prod_{i\geq 1}(1-x^i)^{-A_i} \end{array}$$

The formulas on the last line are derived using the Multiplicity Transform, which realizes a set or multiset S of elements from $\mathcal{A}$ as a multiplicity function $m : \mathcal{A} \to \mathbb{N}$, where $m(a)$ is the number of times a appears in S. Here $n \in \mathbb{N}$ has size $|n| = n$, giving graded bijections $\textsc{Set}\,\mathcal{A} \cong \{0,1\}^{\mathcal{A}}$ and $\textsc{MSet}\,\mathcal{A} \cong \mathbb{N}^{\mathcal{A}}$.

To indicate size, we place the marker x^k next to each element a with $|a| = k$: thus, $\mathbb{N} = \{0, 1, 2, \ldots\} = \{0, 1x, 2x^2, \ldots\}$ with generating function $N(x) = \sum_{k\geq 0} x^k = (1-x)^{-1}$.

Example: Binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ are the counting sequence of $\textsc{Set}([n]x)$, the class of subsets of $[n] = \{1, \ldots, n\}$ with all $|j| = 1$, denoted $[n]x = \{1x, \ldots, nx\}$ so that $|S| = \#S$. Hence the generating function is $\prod_{j\geq 0}(1+x^j)^{A_j} = (1+x)^n$, and Taylor's coefficient formula gives $\binom{n}{k} = n^{\underline{k}}/k!$, where $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$. The identity $(1+x)^n = (1+x)(1+x)^{n-1}$ is equivalent to the Pascal's Triangle recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, which can be proved bijectively by a Deletion Transform taking $S \subset [n]$ with $|S| = k$ on the left to $S' = S \setminus \{n\} \subset [n-1]$ on the right, with $|S'| = k$ or $k-1$.

Multi-choose numbers $\left(\!\binom{n}{k}\!\right)$ count $\textsc{MSet}([n]x)$, multisets of k elements from $[n]$, unordered with repeats allowed, having generating function $(1-x)^{-n}$ and Taylor coefficients $\left(\!\binom{n}{k}\!\right) = n^{\overline{k}}/k!$ where $n^{\overline{k}} = n(n+1)\cdots(n+k-1)$. The Accordion Transform shows $\binom{n}{k} = \left(\!\binom{n-k+1}{k}\!\right)$, taking $S = \{s_1 < s_2 < \cdots < s_k\} \subset [n]$ to a multiset $S^{\downarrow} = \{s_1 \leq s_2-1 \leq \cdots \leq s_k-k+1\}$ from $[n-k+1]$. This is a bijection, which proves the identity.

Example: Fibonacci numbers are defined by the recurrence $F_k = F_{k-1} + F_{k-2}$ starting from $F_0 = 0$, $F_1 = 1$. This implies the generating function equation:

$$F(x) \;=\; \textstyle\sum_{k\geq 0} F_k x^k \;=\; x + \sum_{k\geq 2} F_{k-1}x^k + \sum_{k\geq 2} F_{k-2}x^k \;=\; x + xF(x) + x^2F(x),$$

which can be solved for $F(x) = \frac{x}{1-x-x^2}$. Thus $\frac{1}{x}F(x) = \sum_{k\geq 0} F_{k+1}x^k = \frac{1}{1-(x+x^2)}$ is clearly the generating function of $\mathcal{A} = \textsc{Seq}\{1x, 2x^2\}$, so that:

$$F_{k+1} = \#\mathcal{A}_k = \#\{(a_1, \ldots, a_n) \text{ with } n \geq 0,\ a_j = 1 \text{ or } 2, \text{ and } a_1+\cdots+a_n = k\},$$

the number of compositions of k into parts equal to 1 or 2.

The partial fraction decomposition gives Binet's formula:

$$F(x) = \tfrac{1}{\sqrt{5}}\left(\tfrac{1}{1-\phi x} - \tfrac{1}{1+\psi x}\right) \quad \Longrightarrow \quad F_k = \tfrac{1}{\sqrt{5}}(\phi^k - (-\psi)^k)$$

for the golden ratio $\phi = \frac{\sqrt{5}+1}{2} \approx 1.6$, $\psi = \frac{\sqrt{5}-1}{2} \approx 0.6$. The singularity $x = 1/\phi = \psi$ is closest to the center of the power series $x = 0$, and thus gives the asymptotically largest term: $F_k \sim \frac{1}{\sqrt{5}}\phi^k$ as $k \to \infty$; in fact the error $\pm\frac{1}{\sqrt{5}}\psi^k$ goes to zero, so $F_k = \lfloor \frac{1}{\sqrt{5}}\phi^k \rceil$, the integer rounding of the approximation.

Labeled constructions. A *labeled graded class* $\tilde{\mathcal{A}}$ has objects a with a structure of $|a| = k$ atoms labeled by a bijection with $[k] = \{1, 2, \ldots, k\}$. (Example: graphs on the vertex set $V = [k]$.) Relabeling the atoms gives an action of the symmetric group $\mathfrak{S}_k$ on $\tilde{\mathcal{A}}_k$. The exponential generating function is $\tilde{A}(x) = \sum_{a \in \mathcal{A}} \frac{x^{|a|}}{|a|!} = \sum_{k \geq 0} \tilde{A}_k \frac{x^k}{k!}$.

For $a \in \tilde{\mathcal{A}}_k$ and a set of labels $J = \{j_1 < \cdots < j_k\}$, define a_J to be a relabeled version of a with each atom label i replaced with j_i. We combine labeled classes $\tilde{A}, \tilde{B}$ into the *labeled product* $\tilde{\mathcal{A}} * \tilde{\mathcal{B}}$, consisting of pairs (a_J, b_K) of size $k = |a| + |b|$ in which the label set is partitioned as $[k] = J \sqcup K$ in all possible ways. We also have the directed labeled product $\tilde{\mathcal{A}}^{\min} * \tilde{\mathcal{B}}$, whose elements are (a_J, b_K) with the requirement $1 \in J$. Standard constructions:

$$\begin{array}{llll}
\tilde{\mathcal{C}} = \tilde{\mathcal{A}} * \tilde{\mathcal{B}}, & \tilde{C}(x) = \tilde{A}(x)\tilde{B}(x) & \tilde{\mathcal{C}} = \tilde{\mathcal{A}}^{\min} * \tilde{\mathcal{B}}, & \tilde{C}(x) = \int \tilde{A}'(x)\tilde{B}(x)\,dx \\
\tilde{\mathcal{C}} = \textsc{Seq}^{\sim}_n \tilde{\mathcal{A}}, & \tilde{C}(x) = \tilde{A}(x)^n & \tilde{\mathcal{C}} = \textsc{Seq}^{\sim} \tilde{\mathcal{A}}, & \tilde{C}(x) = 1/(1-\tilde{A}(x)) \\
\tilde{\mathcal{C}} = \textsc{Set}^{\sim}_n \tilde{\mathcal{A}}, & \tilde{C}(x) = \tilde{A}(x)^n/n! & \tilde{\mathcal{C}} = \textsc{Set}^{\sim} \tilde{\mathcal{A}}, & \tilde{C}(x) = \exp \tilde{A}(x) \\
\tilde{\mathcal{C}} = \textsc{Cyc}^{\sim}_n \tilde{\mathcal{A}}, & \tilde{C}(x) = \tilde{A}(x)^n/n & \tilde{\mathcal{C}} = \textsc{Cyc}^{\sim} \tilde{\mathcal{A}}, & \tilde{C}(x) = -\log(1-\tilde{A}(x))
\end{array}$$

There are no labeled multisets: distinct labels prevent repeated parts within an object.

Example: Bell numbers B_k count the class $\tilde{\mathcal{B}}_k$ whose elements are sets $\{J_1, J_2, \ldots,\}$ which partition $[k] = J_1 \sqcup J_2 \sqcup \cdots$ into any number of non-empty subsets $J_i \neq \varnothing$. They are the counting numbers of the class $\tilde{\mathcal{B}} = \textsc{Set}^{\sim}(\textsc{Set}^{\sim}_{\geq 1}([1]x))$, so $\tilde{B}(x) = \exp(e^x - 1) = \frac{1}{e}\sum_{n \geq 0} e^{nx}$, giving Dobinski's formula $B_k = \frac{1}{e}\sum_{n \geq 0} \frac{n^k}{n!}$.

Stirling partition numbers $\left\{{k \atop n}\right\}$ count partitions of $[k]$ into n non-empty subsets, composing the class $\tilde{\mathcal{B}}^{(n)} = \textsc{Set}^{\sim}_n(\textsc{Set}^{\sim}_{\geq 1}([1]x))$ with $\tilde{B}^{(n)}(x) = \frac{1}{n!}(e^x - 1)^n = \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{jx}$:

$$\left\{{k \atop n}\right\} = \tfrac{1}{n!}\left(n^k - \tbinom{n}{1}(n-1)^k + \tbinom{n}{2}(n-2)^k - \cdots + (-1)^{n-1}\tbinom{n}{n-1}1^k\right).$$

Example: Derangement numbers D_k count the class $\tilde{\mathcal{D}}_k$ of derangement permutations $\pi \in \mathfrak{S}_k$ with $\pi(i) \neq i$ for all i, meaning the cycle decomposition of π has no 1-cycles. This can be constructed as $\tilde{\mathcal{D}} = \textsc{Set}^{\sim}(\textsc{Cyc}^{\sim}_{\geq 2}([1]x))$, so $\tilde{D}(x) = \exp(-\log(1-x) - x) = \frac{e^{-x}}{1-x}$, and the derangement number is $D_k = k!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^k\frac{1}{k!}) \approx k!\,e^{-1}$. If k objects are sorted randomly, the chance of no object returning to its former position is about 37%.

Example: Euler numbers E_k count permutations $\pi \in \mathfrak{S}_k$ satisfying the alternating condition: $\pi(1) > \pi(2) < \pi(3) > \cdots$. Let $\tilde{\mathcal{J}} = \coprod_{\ell \geq 0} \tilde{\mathcal{J}}_{2\ell+1}$ be the class of alternating permutations with odd length $k = 2\ell + 1$. A Deletion Transform cuts $\pi = (\pi(1), \ldots, \pi(k))$ at the position j where $\pi(j) = 1$, leaving two smaller odd-length alternating permutations left and right of the cut. This implies the Deletion Recurrence:

$$\tilde{\mathcal{J}} \;\cong\; \tilde{\mathcal{J}} * [1]x^{\min} * \tilde{\mathcal{J}} \;\sqcup\; [1]x, \qquad \tilde{J}(x) \;=\; \int \tilde{J}(x)\tfrac{d}{dx}(x)\tilde{J}(x)\,dx \;+\; x,$$

equivalent to a separable differential equation which integrates to $\tilde{J}(x) = \tan(x)$. That is, the Taylor coefficients of tangent are the odd Euler numbers over $k!$. Similarly, the even-length alternating permutations $\tilde{\mathcal{K}}$ satisfy $\tilde{\mathcal{K}} \cong \tilde{\mathcal{J}} * [1]x^{\min} * \tilde{\mathcal{K}} \ \sqcup \ \{\varnothing\}$, and $\tilde{K}(x) = \sec(x)$.

EXAMPLE: Cayley trees are rooted labeled trees, meaning connected acyclic simple graphs on vertices $V = [k]$, with a distinguished root vertex, composing a labeled class $\tilde{T}_k$. Removing the root gives a Deletion Recurrence:

$$\tilde{\mathcal{T}} \ \cong \ [1]x * \mathrm{SET}^{\sim}(\tilde{\mathcal{T}}), \qquad \tilde{T}(x) = x \exp \tilde{T}(x).$$

That is, $\tilde{T}(x)/\exp \tilde{T}(x) = x$, so $\tilde{T}(x)$ is the inverse function of $A(x) = x/\exp(x)$. The *Lagrange Inversion Formula* states that if $B(x) = x + B_2x^2 + B_3x^3 + \cdots$ is the inverse function of $A(x) = x + A_2x^2 + A_3x^3 + \cdots$, so that $A(B(x)) = x$ as formal power series, then $B_k = \frac{1}{k}[x^{-1}]A(x)^{-k}$, where $[x^{-1}]$ is the operation which extracts the x^{-1} residue coefficient of a Laurent series. So $\tilde{T}_k/k! = \frac{1}{k}[x^{-1}](x/\exp(x))^{-k} = \frac{1}{k}[x^{-1}]\, x^{-k}\exp(kx)$, and $\tilde{T}_k = k^{k-1}$.

This gives Cayley's Theorem that the number of free (non-rooted) trees on k labeled vertices is k^{k-2}. This can also be proved bijectively by Prüfer's bijection, which takes a free tree T on vertices $V = [k]$, and removes its minimal-label leaf vertex ℓ_1, while recording ℓ_1's unique neighbor vertex a_1; repeating recursively gives the Prüfer sequence $(a_1, \ldots, a_{n-2}) \in [k]^{k-2}$, from which T can be reconstructed via $\ell_1 = \min([k] \smallsetminus \{a_1, \ldots, a_{n-2}\})$, etc. Alternatively, Joyal's bijection takes a birooted tree (T, u, v) for $u, v \in V = [k]$, with a unique "spine" path $u = b_1' — b_2' — \cdots — b_m' = v$, and orders these vertices as $B = \{b_1', \ldots, b_m'\} = \{b_1 < \cdots < b_m\}$; it also orients each non-spine edge toward the spine, $w \to w'$ for $w \notin B$; then it defines a function $f : [k] \to [k]$ by $f(b_i) = b_i'$, viewing the spine as a permutation of B in one-line notation, and $f(w) = w'$ for $w \notin B$. Then (T, u, v) is equivalent to f, and the number of birooted trees is k^k.

Bigraded constructions. A *bigraded class* $\mathcal{A}$ is a class whose objects are given two measures of magnitude, size $|a| = k$ and weight $\mathrm{wt}(a) = n$, with counting numbers $A_k^{(n)} = \#\{a \in \mathcal{A} \text{ with } |a| = k,\ \mathrm{wt}(a) = n\}$ and bivariate generating function $\mathcal{A}(x, y) = \sum_{a \in \mathcal{A}} x^{|a|} y^{\mathrm{wt}(a)} = \sum_{k,n \geq 0} A_k^{(n)} x^k y^n$. In a *labeled bigraded class* $\tilde{\mathcal{A}}$, each object with $|a| = k$ is labeled with $[k]$, and the permutation action of $\mathfrak{S}_k$ preserves both $|a|$ and $\mathrm{wt}(a)$. We have the exponential bivariate generating function $\tilde{A}(x, y) = \sum_{a \in \tilde{\mathcal{A}}} \frac{x^{|a|}}{|a|!}\, y^{\mathrm{wt}(a)} = \sum_{n,k \geq 0} \tilde{A}_k^{(n)} \frac{x^k}{k!}\, y^n$.

The constructions for unlabeled and labeled graded classes extend to the bigraded case. We again indicate magnitudes on the resulting class by inserting markers: x^k for unlabeled size, $\frac{x^k}{k!}$ for labeled size, and y^n for unlabeled weight:

$$\tilde{\mathcal{A}}_k \tfrac{x^k}{k!}, \quad \tilde{\mathcal{A}}(x) = \textstyle\coprod_{k \geq 0} \tilde{\mathcal{A}}_k \tfrac{x^k}{k!}, \quad \mathcal{B}_n\, y^n, \quad \mathcal{B}(y) = \textstyle\coprod_{n \geq 0} \mathcal{B}_n y^n, \quad \text{etc.}$$

Thus $\tilde{\mathcal{C}}(x, y) = \tilde{\mathcal{A}}(x) \times \mathcal{B}(y)$ means the bigraded class of (a, b) with a labeled, b unlabeled, size $|(a, b)| = |a| = k$, weight $\mathrm{wt}(a, b) = |b| = n$, and $\tilde{C}(x, y) = \tilde{A}(x)B(y)$. A different product is $\tilde{\mathcal{A}}(x, y) * \tilde{\mathcal{B}}(x, y)$, meaning relabeled pairs (a_J, b_K) with $|(a_J, b_K)| = |a| + |b|$ and $\mathrm{wt}(a_J, b_K) = \mathrm{wt}(a) + \mathrm{wt}(b)$, giving the generating function $\tilde{A}(x, y)\tilde{B}(x, y)$.

We also have the atomic function construction from $\tilde{\mathcal{A}}$ to $\mathcal{B}$, the class of functions from the atoms of some $[k]$-labeled $a \in \tilde{\mathcal{A}}$ to $\mathcal{B}$:

$$\tilde{\mathcal{C}}(x, y) = \mathrm{AFun}(\tilde{\mathcal{A}}(x), \mathcal{B}(y)) = \{(a, f) \text{ with } a \in \mathcal{A},\ f \colon [k] \to \mathcal{B}\}, \quad \tilde{C}(x, y) = \tilde{A}(x\mathcal{B}(y));$$

here $|(a, f)| = |a| = k$ is marked by $\frac{x^k}{k!}$, and $\mathrm{wt}(a, f) = \sum_{i=1}^{k} |f(i)| = n$ is marked by y^n.

Example: Binomial coefficient identities. Consider the unlabeled bigraded class of subsets inside an integer interval, pairs $(S{\subset}[n])$, with size function $|(S{\subset}[n])| = \#S = k$ marked by x^k, and weight function $\mathrm{wt}(S{\subset}[n]) = n$ marked by y^n. Its bivariate generating function is:

$$A(x,y) = \textstyle\sum_{n,k\geq 0}\binom{n}{k}x^k y^n = \sum_{n\geq 0} y^n(1{+}x)^n = \frac{1}{1-y(1+x)} = \sum_{k\geq 0} A_k(y)x^k,$$

where $A_k(y) = \sum_{n\geq 0}\binom{n}{k}y^n$. Applying Taylor's coefficient formula to the variable x gives $A_k(y) = \frac{\partial^k}{\partial x^k}A(x,y)|_{x=0} = \frac{y^k}{(1-y)^{k+1}}$. The algebraic identity $\frac{1}{1-y}A_k(y) = \frac{1}{y}A_{k+1}(y)$ is equivalent to the combinatorial identity:

$$\tbinom{k}{k} + \tbinom{k+1}{k} + \cdots + \tbinom{n}{k} = \tbinom{n+1}{k+1}.$$

This can be proved bijectively by a Deletion Transform which takes a $(k{+}1)$-element subset $S\subset[n{+}1]$ on the right side to the k-element subset $S' = S\setminus\{m\}\subset[m{-}1]$ on the left side, where and $m = \max(S)$.

Furthermore, substituting $x = z$, $y = z$ reduces to the single grading $\|(S{\subset}[n])\| = \#S + n = \ell$ marked by z^ℓ. This has generating function $A(z,z) = \frac{1}{1-z(1+z)} = \frac{1}{1-z-z^2}$, which we recognize as the Fibonacci generating function $\sum_{\ell\geq 0}F_{\ell+1}z^\ell$. This shows:

$$F_{\ell+1} \;=\; \sum_{n+k=\ell}\tbinom{n}{k} \;=\; \tbinom{\ell}{0} + \tbinom{\ell-1}{1} + \tbinom{\ell-2}{2} + \cdots .$$

To prove bijectively: let $F_{\ell+1} = \#\{\, a = (a_1,\ldots,a_n) \mid n\geq 0,\ a_i = 1 \text{ or } 2,\ \sum_{i=1}^n a_i = \ell\,\}$, and transform the composition a to the set $S = [\ell]\setminus\{a_1, a_1{+}a_2, , \ldots, a_1{+}\cdots{+}a_n\}\subset[\ell{-}1]$, then apply the Accordion Transform to get $S^{\downarrow}\subset[n]$, with $|S^{\downarrow}| = \ell{-}n$.

The bigraded class of multisets with k elements from $[n]$ has generating function:

$$B(x,y) = \sum_{n,k\geq 0}\left(\!\!\binom{n}{k}\!\!\right)x^k y^n = \sum_{n\geq 0}\frac{1}{(1-x)^n}y^n = \frac{1}{1-\frac{y}{1-x}} = 1 + \frac{y}{1-x-y}.$$

The algebraic identity $A(x,y) = \frac{1}{y}(B(xy,y)-1)$ means that $\binom{n}{k} = \left(\!\binom{m}{k}\!\right)$ for $n = m{+}k{-}1$, recovering the Accordion Transformation identity $\binom{n}{k} = \left(\!\binom{n+k-1}{k}\!\right)$.

Example: Stirling cycle numbers. Let $\tilde{\mathcal{S}}(x)$ be the union of the finite symmetric groups $\tilde{\mathcal{S}}_k = \mathfrak{S}_k$ for $k\geq 1$ and $\tilde{S}_0 = \{\varnothing\}$, thought of as the labeled class of all permutations $\pi : [k] \xrightarrow{\sim} [k]$, with size function $|\pi| = k$. This is constructed as $\tilde{\mathcal{S}}(x) = \mathrm{Seq}^{\sim}([1]x)$ with generating function $\tilde{S}(x) = (1{-}x)^{-1}$. But permutations are also sets of labeled cycles, so we enrich this to a bigraded labeled class $\tilde{\mathcal{S}}(x,y)$ with $\mathrm{wt}(\pi) = \mathrm{cyc}(\pi) = n$, the number of disjoint cycles composing π, marked by y^n. The counting numbers are Stirling cycle numbers:

$$\begin{bmatrix}k\\n\end{bmatrix} = \tilde{S}_k^{(n)} = \#\{\pi\in\mathfrak{S}_k \text{ with } \mathrm{cyc}(\pi) = n\}.$$

We construct:

$$\tilde{\mathcal{S}}(x,y) = \mathrm{Set}^{\sim}(y\,\mathrm{Cyc}^{\sim}([1]x)), \qquad \tilde{S}(x,y) = \exp(-y\log(1{-}x)) = (1{-}x)^{-y}.$$

Setting $y = 1$ recovers $\tilde{S}(x) = \tilde{S}(x,y)|_{y=1}$. Taking the coefficient of $\frac{x^k}{k!}$ in $\tilde{S}(x,y)$:

$$S_k(y) \;=\; \textstyle\sum_{n=1}^k \begin{bmatrix}k\\n\end{bmatrix} y^n \;=\; \frac{\partial^k}{\partial x^k}\tilde{S}(x,y)|_{x=0} \;=\; y^{\overline{k}}.$$

Substituting $y \mapsto -y$ turns this into $\sum_{n=1}^{k}(-1)^{k-n}\left[{k \atop n}\right]y^n = y^{\underline{k}}$.

The average number of cycles $\mathrm{cyc}(\pi)$ of a permutation $\pi \in \mathfrak{S}_k$ is a Harmonic number:

$$\begin{aligned}\tfrac{1}{k!}\textstyle\sum_{n=1}^{k}\left[{k \atop n}\right]n &= \tfrac{1}{k!}\tfrac{\partial}{\partial y}S_k(y)|_{y=1} = [x^k]\tfrac{\partial}{\partial y}S(x,y)|_{y=1} = [x^k]\tfrac{\partial}{\partial y}(1-x)^{-y}|_{y=1} \\ &= [x^k]\tfrac{\log(1-x)}{1-x} = [x^k](\sum_{i\geq 1}\tfrac{x^i}{i}\sum_{j\geq 0}x^j) = \sum_{j=1}^{k}\tfrac{1}{j} \approx \log(k),\end{aligned}$$

where $[x^k]$ is the operation which extracts the x^k coefficient of a power series. It can also be shown that the standard deviation of $\mathrm{cyc}(\pi)$ approaches $\sqrt{\log(k)}$, so for large k, almost all permutations in $\mathfrak{S}_k$ have approximately $\log(k)$ cycles.

The other one-variable generating function is:

$$\tilde{S}^{(n)}(x) = \textstyle\sum_{k\geq 1}\left[{k \atop n}\right]\frac{x^k}{k!} = \log^n(\frac{1}{1-x}).$$

This does not give an explicit formula for the coefficients, but complex analytic methods can produce the asymptotic approximation: $\left[{k \atop n}\right] \sim \frac{(k-1)!}{(n-1)!}(\log k)^{n-1}$ as $k \to \infty$. Thus for large k and fixed n, the fraction of k-permutations with n cycles is very close to $\frac{(\log k)^{n-1}}{k(n-1)!}$.

EXAMPLE: Partition numbers. A partition of k is a set of non-negative integers

$$\lambda = \{\lambda_1 \geq \cdots \geq \lambda_n\} \qquad \text{with} \qquad |\lambda| = \lambda_1 + \cdots + \lambda_n = k,$$

allowing $\lambda_i = 0$. Its length is $\ell(\lambda) = n$. By tradition, $|\lambda| = k$ is marked by q^k, while $\ell(\lambda) = n$ is marked by x^n, making an unlabeled bigraded class ${}^\circ\mathcal{P}(q,x)$ whose counting sequence is denoted $p_n(k) = {}^\circ P_k^{(n)}$. The $\circ$ superscript indicates that we allow zero parts $\lambda_i = 0$; the subclass with all $\lambda_i \geq 1$ is denoted $\mathcal{P}(q,x)$.

The Multiplicity Transform turns λ into $m : \{0, 1q, 2q^2, \ldots\} \to \{0, 1qx, 2q^2x^2, \ldots\}$ with $m(j) = \#\{i \text{ with } \lambda_i = j\}$, so that $|\lambda| = \sum_{j\geq 0} j\, m(j)$ and $\ell(\lambda) = \sum_{j\geq 0} m(j)$. This gives the generating function, often written in terms of the q-Pochhammer symbol $(x;q)_n = (1-x)(1-qx)(1-q^2x)\cdots(1-q^{n-1}x)$:

$${}^\circ P(q,x) = \sum_{k,n\geq 0} p_n(k)\, q^k x^n = \prod_{j\geq 0}\frac{1}{1-q^j x} = \frac{1}{(x;q)_\infty}.$$

Another picture of λ is its Ferrars diagram: n left-justified rows of spaced dots, with successive row lengths $\lambda_1, \ldots, \lambda_n$. Reflecting across the main diagonal gives the Transpose Transform $\lambda \mapsto \lambda'$ defined by $\lambda'_j = \#\{i \text{ with } \lambda_i \geq j\} = m(j) + m(j+1) + \cdots$, with $|\lambda'| = |\lambda|$ and indeterminate $\ell(\lambda')$. This gives a bijection between all λ with $\ell(\lambda) = n$ parts and all λ' with each part $\lambda'_j \leq n$ and indeterminate $\ell(\lambda')$. Counting $m(\lambda')$ over $\lambda \in {}^\circ\mathcal{P}^{(n)}(q)$ gives:

$${}^\circ P(q,x) = e_q(x) \overset{\text{def}}{=} \sum_{n\geq 0}\frac{x^n}{(q)_n} \qquad \text{where} \qquad (q)_n = (q;q)_n = (1-q)\cdots(1-q^n).$$

This is one version of the q-Binomial Theorem. The notation $e_q(x)$ is meant to suggest a q-analog of the exponential function: indeed, $\lim_{q\to 1}(q)_n/(1-q)^n = n!$, and $\lim_{q\to 1} e_q((1-q)x) = \sum_{n\geq 0}\frac{x^n}{n!} = \exp(x)$.

An important subclass $^\circ\mathcal{Q}(q,x) \subset {^\circ\mathcal{P}}(q,x)$ comprises partitions with *distinct* parts: $\mu = \{\mu_1 > \cdots > \mu_n\}$. Counting $m(\mu)$ gives $^\circ Q(q,x) = \prod_{j\geq 0}(1+q^j x)$. The Accordion Transform $\mu \mapsto {^\downarrow\mu}$ takes μ with n distinct parts to a general partition $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_n\}$:

$$\lambda = {^\downarrow\mu} = \{\mu_1{-}n{+}1 \geq \mu_2{-}n{+}2 \geq \cdots \geq \mu_{n-1}{-}1 \geq \mu_n\},$$

reducing $|\mu|$ by $(n{-}1)+(n{-}2)+\cdots+1 = \binom{n}{2}$. Counting $m(\lambda') = m({^\downarrow\mu'})$ for each n gives:

$$^\circ Q(q,x) \;=\; \prod_{j\geq 0}(1{+}q^j x) \;=\; E_q(x) \;\stackrel{\text{def}}{=}\; \sum_{n\geq 0} q^{\binom{n}{2}}\frac{x^n}{(q)_n}.$$

Note the algebraic identities $e_q(x)E_q(-x) = 1$ and $E_q(x) = e_{1/q}(-x/q)$.*

If we forget the number of parts in $^\circ\mathcal{P}(q,x)$, the q^k coefficients become infinite, so we must allow only partitions with positive parts, $\lambda \in \mathcal{P}(q)$ with counting sequence $p(k) = p_k(k)$ and generating function $P(q) = (1{-}x)P(q,x)|_{x=1}$. Similarly for $\mathcal{Q}$, the partitions with distinct parts. Thus:

$$P(q) = \textstyle\prod_{j\geq 1}\frac{1}{1-q^j}, \qquad Q(q) = \prod_{j\geq 1}(1{+}q^j) = \frac{1}{P(-q)}.$$

This produces Dedekind's eta function using $q = e^{2\pi i\tau}$, the nome: $\eta(\tau) = q^{1/24}/P(q) = q^{1/24}Q(-q)$ is a weight $\frac{1}{2}$ modular form with $\eta(\tau{+}1) = \eta(\tau)$ and $\eta(-1/\tau) = -\sqrt{i\tau}\,\eta(\tau)$.

There is no closed combinatorial formula to compute $p(k)$, but complex analysis applied to $P(q)$, which is singular at every complex root of unity, yields the celebrated Hardy-Ramanujan asymptotic $p(k) \sim \frac{1}{4k\sqrt{3}}\exp(\pi\sqrt{2k/3})$.

Signed constructions. A *signed graded class* $\mathcal{A} = \mathcal{A}^+ \sqcup \mathcal{A}^-$ is a class with a function $\mathrm{sgn} : \mathcal{A} \to \{\pm 1\}$, counted by $A_k^+ = \#\mathcal{A}_k^+$, $A_k^- = \#\mathcal{A}_k^-$, and signed generating function:

$$A^\pm(x) \;=\; \textstyle\sum_{a\in A}\mathrm{sgn}(a)x^{|a|} \;=\; \sum_{k\geq 0}(A_k^+{-}A_k^-)x^k.$$

Suppose we have an involution, $I : \mathcal{A} \to \mathcal{A}$ with $I^{-1} = I$, which preserves size, $|I(a)| = |a|$, and which reverses sign: $\mathrm{sgn}(I(a)) = -\mathrm{sgn}(a)$, except when $I(a) = a$.

Involution Priniciple: The signed generating function of $\mathcal{A}$ is equal to that of the fixed point set $\mathcal{F} = \mathcal{A}^I = \{a \in \mathcal{A} \mid I(a) = a\}$, since each non-fixed a is canceled by $I(a)$:

$$\mathcal{F} = \mathcal{A}^I \quad\Longrightarrow\quad F^\pm(x) = A^\pm(x).$$

Example: The Principle of Inclusion-Exclusion results from a Max-Transfer Involution. Given a class $\mathcal{A}$ and $\mathcal{B}_1,\ldots,\mathcal{B}_n \subset \mathcal{A}$, let $\mathcal{C} = \{(J,a) \in \mathrm{Set}[n]\times\mathcal{A} : a \in \mathcal{B}_J\}$, where $\mathcal{B}_J = \cap_{j\in J}\mathcal{B}_j$ and $\mathcal{B}_\varnothing = \mathcal{A}$. Let $|(J,a)| = |a|$ and $\mathrm{sgn}(J,a) = (-1)^{\#J}$. For $j(a) = \max\{j \in [n] : a \in \mathcal{B}_j\}$, define an involution by:

$$I(J,a) = (J',a) \quad \text{where} \quad J' \;=\; \begin{cases} J\smallsetminus\{j(a)\} & \text{if } j(a)\in J,\\ J\cup\{j(a)\} & \text{if } j(a)\notin J,\\ J=\varnothing & \text{if } \nexists j(a) \text{ since } a\notin\mathcal{B}_j \text{ for all } j.\end{cases}$$

This involution gives the Principle of Inclusion-Exclusion:

$$\mathcal{G} \;\stackrel{\text{def}}{=}\; \mathcal{A}\smallsetminus\cup_{j=1}^n\mathcal{B}_j \;\cong\; (\mathcal{C}^\pm)^I, \qquad G(x) = C^\pm(x) = \textstyle\sum_{J\subset[n]}(-1)^{\#J}B_J(x).$$

If $\mathcal{A}$ is finite, we get the usual formula: $\#(\mathcal{A}\smallsetminus\cup_{j=1}^n\mathcal{B}_j) = \sum_{J\subset[n]}(-1)^{\#J}\#(\cap_{j\in J}\mathcal{B}_j)$.

*The first formula $^\circ P(q,x)\,{^\circ Q}(q,-x) = 1$ can be proved by a Min-Transfer Involution on $(\lambda,\mu) \in {^\circ\mathcal{P}}\times{^\circ\mathcal{Q}}$.

EXAMPLE: *Euler's Pentagonal Number Theorem.* This expands the product formula for $Q(q)$, the generating function of partitions with distinct parts:

$$Q(-q) \;=\; \frac{1}{P(q)} \;=\; \prod_{j\geq 1}(1-q^j) \;=\; 1+\sum_{n\geq 1}(-1)^n\left(q^{n(3n-1)/2}+q^{n(3n+1)/2}\right).$$

Then $P(q)Q(-q)=1$ is equivalent to the weird recurrence:

$$p(k) \;=\; \textstyle\sum_{n\geq 1}(-1)^{n-1}(\,p(k-\frac{1}{2}n(3n-1))+p(k-\frac{1}{2}n(3n+1)\,),$$

where we take $p(0)=1$ and $p(k)=0$ for $k<0$.

The theorem can be proved using Franklin's Involution, a kind of Min-Diagonal Transfer. The left side $Q(-q)$ is the signed generating function for $\mu_1>\cdots>\mu_n>0$ endowed with $\mathrm{sgn}(\mu)=(-1)^n$, the parity of the number of positive parts. In the Ferrars diagram of μ, let $\ell=\mu_n$ be the length of the lowest row, and $d=\max\{i\mid \mu_i=\mu_1-i+1\}$ the length of the diagonal of slope -1 along the top right edge of the diagram. Define a sign-reversing, size-preserving involution:

$$I(\mu_1,\ldots,\mu_n) \;=\; \begin{cases} (\mu_1,\ldots,\mu_n) & \text{if } \mu \text{ is a pentagonal partition,}\\ (\mu_1{+}1,\ldots,\mu_\ell{+}1,\mu_{\ell+1},\ldots,\mu_{n-1}) & \text{otherwise if } \ell\leq d,\\ (\mu_1{-}1,\ldots,\mu_d{-}1,\mu_{d+1},\ldots,\mu_n,d) & \text{otherwise if } \ell>d.\end{cases}$$

The pentagonal partitions are of the form $\mu=(2n{-}1,2n{-}2,\ldots,n)$ with $|\mu|=\frac{1}{2}n(3n{-}1)$, and $\mu=(2n,2n{-}1,\ldots,n{+}1)$ with $|\mu|=\frac{1}{2}n(3n{+}1)$, the only μ for which the manipulations on the second or third lines will not yield a valid partition. The involution I matches pairs of partitions which cancel in the signed generating function, leaving only the pentagonal partitions uncanceled on the right side of the equation, proving its validity.

EXAMPLE: Catalan numbers C_k count the class $\mathcal{C}$ of Dyck paths, which are sequences $\epsilon=(\epsilon_1,\ldots,\epsilon_{2k})$ with all $\epsilon_i=\pm 1$, $\epsilon_1+\cdots+\epsilon_i\geq 0$, and $\epsilon_1+\cdots+\epsilon_{2k}=0$. We may think of ϵ as the win/loss record of $2k$ unit bets, with the requirements that cumulative winnings never dip into bankruptcy and break even at the end. We can split ϵ by removing $\epsilon_1=1$ and the first step $\epsilon_{2i}=-1$ with $\epsilon_1+\cdots+\epsilon_{2i}=0$. This breaks $\epsilon\in\mathcal{C}_k$ into left and right parts $\mathcal{C}_{i-1}\times\mathcal{C}_{k-i}$, leading to the Deletion Recurrence $\mathcal{C}\cong\mathcal{C}{\times}\{\bullet\}{\times}\mathcal{C}\;\sqcup\;\{\varnothing\}$ and the generating function identity $C(x)=xC(x)^2+1$, giving $C(x)=\frac{1-\sqrt{1-4x}}{2x}$. We see $x\,C(x)$ is the inverse function of $A(x)=x(1{-}x)$, so Lagrange Inversion gives: $C_{k+1}=\frac{1}{k}[x^{-1}]A(x)^{-k}=\frac{1}{k}[x^{k-1}](1{-}x)^{-k}=\frac{1}{k}\left(\!\binom{k}{k-1}\!\right)=\frac{1}{k}\binom{2k-2}{k-1}$, so $C_k=\frac{1}{k+1}\binom{2k}{k}$.

We get another formla for C_k using a Path Reflection Involution. Let

$$\mathcal{B}_k^+=\{\epsilon\in\{\pm1\}^{2k}\mid \textstyle\sum_{i=1}^{2k}\epsilon_i=0\},\qquad \mathcal{B}_k^-=\{\epsilon\in\{\pm1\}^{2k}\mid \textstyle\sum_{i=1}^{2k}\epsilon_i=-2\}.$$

For $\epsilon\in\mathcal{C}_k\subset\mathcal{B}_k^+$, define $I(\epsilon)=\epsilon$; any other $\epsilon\in\mathcal{B}_k^\pm$ has a minimal i with $\sum_{j=1}^i\epsilon_j=-1$, and we define $I(\epsilon)=(\epsilon_1,\ldots,\epsilon_i,-\epsilon_{i+1},\ldots,-\epsilon_{2k})$. This pairs $\mathcal{B}_k^+\smallsetminus\mathcal{C}_k$ with $\mathcal{B}_k^-$, showing that $C_k=B_k^+-B_k^-=\binom{2k}{k}-\binom{2k}{k-1}$.

EXAMPLE: Stirling numbers and Lonely/Crowded Involution. The formulas

$$\textstyle\sum_{n=1}^k {k \brace n} y^{\underline{n}}=y^k,\qquad \sum_{n=1}^k(-1)^{k-n}{k \brack n}y^n=y^{\underline{k}}$$

express that the Stirling partition and cycle numbers are change-of-basis coefficients for the polynomial ring $\mathbb{C}[y]$, between the standard basis $\{y^k\}_{k\geq 0}$ and the falling-power basis $\{y^{\underline{k}}\}_{k\geq 0}$. This implies the infinite lower-triangular matrices $M_1 = [\, \left\{ {k \atop n} \right\}]_{k,n\geq 1}$ and $M_2 = [\,(-1)^{k-n} \left[{k \atop n} \right]]_{k,n\geq 1}$ are inverse to each other: $M_1 \cdot M_2 = \mathrm{Id}$. That is, for all $k, n \geq 1$,

$$\sum_{j\geq 1} (-1)^{j-n} \left\{ {k \atop j} \right\} \left[{j \atop n} \right] \;=\; \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

This formula can be proved combinatorially using the Lonely/Crowded Involution. The left side counts permuted set partitions (S, π): an unordered partition $S = \{S_1, \ldots, S_j\}$ with $S_1 \sqcup \cdots \sqcup S_j = [k]$ and $S_i \neq \varnothing$, numbered lexicographically so that $\min(S_1) < \cdots < \min(S_j)$, along with a permutation $\pi \in \mathfrak{S}_j$ with n cycles. Also $\mathrm{sgn}(S, \pi) = (-1)^{j-n}$. In the signed count $\sum_{j\geq 0} (-1)^{j-n} \left\{ {k \atop j} \right\} \left[{j \atop n} \right]$, the involution I will pair up and cancel all terms except a single fixed point, giving the right side.

The involution will define $I(S, \pi) = (S', \pi')$. For $\ell \in [n]$, form the union of the cycle of sets $S(\ell) \overset{\text{def}}{=} S_i \cup S_{\pi(i)} \cup S_{\pi(\pi(i))} \cup \cdots$, where $\ell \in S_i$. Take the smallest ℓ such that $\#S(\ell) \geq 2$. If $S_i = \{\ell\}$ is a singleton, then join it with the next set on its cycle: $S'_i = S_i \cup S_{\pi(i)}$ and $\pi'(i) = \pi(\pi(i))$. If S_i is not a singleton, split it into two sets along the same cycle: $S'_i = \{\ell\}$ and $S'_{\pi'(i)} = S_i - \{\ell\}$, with $\pi'(\pi'(i)) = \pi(i)$. This changes the sign $(-1)^{j-n}$ by incrementing/decrementing j while leaving k, n fixed. If there is no such ℓ, then this is the unique fixed point with k singleton sets $S = \{\{1\}, \ldots, \{k\}\}$ and $\pi = \mathrm{id}$.

Quotient constructions. We say that a group G acts on a set $\mathcal{A}$ if each $g \in G$ gives a bijection $g : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$, and $g(h(a)) = (g \cdot h)(a)$ for all $g, h \in G$. An orbit is a set $G(a) = \{g(a) \mid g \in G\}$.

Burnside Theorem: For a group G acting on a finite set $\mathcal{A}$, the set of orbits

$$\bar{\mathcal{A}} \;=\; \mathcal{A}/G \;\overset{\text{def}}{=}\; \{G(a) \mid a \in \mathcal{A}\}$$

is counted by the average number of fixed points $\mathcal{A}^g = \{a \in \mathcal{A} \mid g(a) \,=\, a\}$:

$$\#\bar{\mathcal{A}} \;=\; \frac{1}{\#G} \sum_{g\in G} \#\mathcal{A}^g\,.$$

Proof: The size of an orbit is $\#G(a) = \#G/\#\mathrm{Stab}(a)$, where $\mathrm{Stab}(a) = \{g \in G \mid g(a) = a\}$. Let $\mathcal{S} = \{(g, a) \in G \times \mathcal{A} \mid g(a) = a\}$, counting $\#\mathcal{S} = \sum_{a\in\mathcal{A}} \#\mathrm{Stab}(a) = \sum_{g\in G} \#\mathcal{A}^g$. Thus:

$$\#\bar{\mathcal{A}} \;=\; \sum_{a\in\mathcal{A}} \frac{1}{\#G(a)} \;=\; \sum_{a\in\mathcal{A}} \frac{\#\mathrm{Stab}(a)}{\#G} \;=\; \frac{\#\mathcal{S}}{\#G} \;=\; \frac{1}{\#G} \sum_{g\in G} \#\mathcal{A}^g\,.$$

If $\mathcal{A}$ is graded and the same G acts on each $\mathcal{A}_k$, we get the generating function formula:

$$\bar{A}(x) \;=\; \frac{1}{\#G} \sum_{g\in G} A^g(x).$$

Example: Necklace polynomials. Consider the class of colored necklaces $\mathcal{N} = \{f : [k] \to [n]\}$; we picture f as a string of k beads chosen from n colors, with the action of the cyclic symmetry group $G = C_k \subset \mathfrak{S}_k$ generated by the rotation $\rho = (12\cdots k)$. A function $f \in \mathcal{N}$ is a fixed point of π if it has a constant value $f(i)$ for all i within a cycle of π. The rotation $\pi = \rho^j$ has $d = \gcd(j,k)$ cycles of length k/d, and each cycle has choice of n colors, so $\#\mathcal{N}^\pi = n^d$. Since there are $\varphi(k/d)$ such rotations, where φ is Euler's totient function, the orbits (distinct necklaces) are counted by the *necklace polynomial*:

$$N_k(n) \;=\; \#(\mathcal{N}/C_k) \;=\; \tfrac{1}{k}\textstyle\sum_{\pi\in\mathcal{C}_k} \#\mathcal{N}^\pi \;=\; \tfrac{1}{k}\sum_{d|k} \varphi(k/d)\, n^d.$$

This has a remarkable alternative meaning. In the finite field $\mathbb{F}_q$ for a prime power $q = p^k$, the Galois group over the prime field $\mathbb{F}_p$ is the cyclic group C_k generated by the Frobenius automorphism $\Phi(\alpha) = \alpha^p$. Writing field elements in terms of a Galois normal basis $B = \{\gamma, \Phi(\gamma), \ldots, \Phi^{k-1}(\gamma)\}$ over the prime field $\mathbb{F}_p$ makes each element of $\mathbb{F}_q$ correspond to a coefficient function $f : B \to \mathbb{F}_p$, equivalent to $f : [k] \to [p]$ with the cyclic C_k action. Hence the number of distinct necklaces $N_k(p)$ is equal to the number Galois orbits on $\mathbb{F}_q$, which Galois theory shows to be the number of irreducible monic polynomials in $\mathbb{F}_p[x]$ of all degrees d dividing k. For example, $N_3(n) = \frac{1}{3}(n^3{+}2n)$, and in $\mathbb{F}_2[x]$ there are $N_3(2) = 4$ irreducible monic polynomials of degree 3 or 1: $x^3{+}x{+}1$, $x^3{+}x^2{+}1$, x, $x{+}1$.

Polya's Method. We refine the above to keep track of the number of beads with each of the n colors, marking them with variables $y_1, \ldots, y_n$. Thus we consider a multi-graded class of functions having n weight measures:

$$\mathcal{F}(\vec{y}) = \mathcal{F}(y_1, \ldots, y_n) = \{f : [k] \to [n]\}, \quad \mathrm{wt}_1(f) = \#f^{-1}(1), \ldots, \mathrm{wt}_n(f) = \#f^{-1}(n).$$

This has multivariate generating function:

$$F(\vec{y}) \;=\; \textstyle\sum_{f\in\mathcal{F}} y_1^{\mathrm{wt}_1(f)} \cdots y_n^{\mathrm{wt}_n(f)} \;=\; \displaystyle\sum_{k_1+\cdots+k_n=k} F_{k_1,\ldots,k_n}\, y_1^{k_1} \cdots y_n^{k_n}.$$

A permutation group $G \subset \mathfrak{S}_k$ induces an action on $f \in \mathcal{F}$ by $\pi(f)(i) = f(\pi^{-1}(i))$, where the inverse π^{-1} is needed to get $\pi_1(\pi_2(f)) = (\pi_1 \cdot \pi_2)(f)$. The quotient class of orbits $\bar{\mathcal{F}} = \mathcal{F}/G$ has generating function $\bar{F}(\vec{y})$, called *Polya's pattern inventory polynomial* $P_G(\vec{y})$.

To apply Burnside's Theorem, we must count the fixed class $\mathcal{F}^\pi$, consisting of the functions which are constant on each cycle of π. For such functions, each i-cycle of π contributes a choice of i identical colors with generating function $p_i(\vec{y}) = y_1^i + \cdots + y_n^i$. So:

$$F^\pi(\vec{y}) \;=\; p_1(\vec{y})^{\mathrm{cyc}_1(\pi)} \cdots p_k(\vec{y})^{\mathrm{cyc}_k(\pi)},$$

where $\mathrm{cyc}_i(\pi) = \#$ i-cycles of π acting on $[k]$. Then Burnside's Theorem gives:

$$\bar{F}(\vec{y}) \;=\; \frac{1}{\#G} \sum_{\pi} \bar{F}^\pi(\vec{y}) \;=\; \frac{1}{\#G} \sum_{\pi\in G} p_1(\vec{y})^{\mathrm{cyc}_1(\pi)} \cdots p_k(\vec{y})^{\mathrm{cyc}_k(\pi)}.$$

In Polya's notation: $\bar{F}(\vec{y}) \;=\; P_G(\vec{y}) \;=\; \frac{1}{\#G} Z_G(p_1(\vec{y}), \ldots, p_k(\vec{y}))$, where

$$Z_G(z_1, \ldots, z_k) \;=\; \textstyle\sum_{\pi\in G} z_1^{\mathrm{cyc}_1(\pi)} \cdots z_k^{\mathrm{cyc}_k(\pi)}$$

is called the *cycle index polynomial*; for example $Z_{\mathfrak{S}_3}(z_1, z_2, z_3) = z_1^3 + 3z_1z_2 + 2z_3$.

Example: Reconsidering necklaces, the cyclic group $G = C_k \subset \mathfrak{S}_k$ has cycle index:

$$Z_G(z_1, \ldots, z_k) \;=\; \textstyle\sum_{d|k} \varphi(k/d) z_{k/d}^d \;=\; \sum_{d|k} \varphi(d) z_d^{k/d}.$$

Hence the color-counting necklace function $N_k(\vec{y}) = P_{C_k}(\vec{y}) = \bar{F}(\vec{y})$ is:

$$N_k(y_1,\ldots,y_n) \;=\; \tfrac{1}{k}\sum_{d|k}\varphi(d)\,(y_1^d+\cdots+y_n^d)^{k/d}.$$

The necklace polynomial $N_k(n)$ is the specialization of $N_k(y_1,\ldots,y_n)$ at $y_1=\cdots=y_n=1$, so that $y_1^d+\cdots+y_n^d=n$.

Taking $k=6$ beads with $n=2$ colors (red, blue) marked by $y_1=r$, $y_2=b$, we have:

$$\begin{aligned} N_6(r,b) &= \tfrac{1}{6}((r+b)^6+(r^2+b^2)^3+2(r^3+b^3)^2+2(r^6+b^6)) \\ &= b^6+rb^5+3r^2b^4+4r^3b^3+3r^4b^2+r^5b+r^6. \end{aligned}$$

The coefficient of r^3b^3 counts the 4 necklaces: $rrrbbb, rrbrrb, rrbbrb, rbrbrb$. The necklace polynomial is $N_6(n) = N_6(1,\ldots,1) = \frac{1}{6}(n^6+n^3+2n^2+2n)$, so there are $N_6(2)=14$ necklaces with 6 beads and any frequencies of r,b.

Multisets from a class. For a general unlabeled class $\mathcal{B}(y)$ with $\mathcal{B}_0=\{\}$, consider the unlabeled bigraded class of multi-sets of $\mathcal{B}$:

$$\mathcal{M}(x,y)=\mathrm{MS{\scriptsize ET}}(x\mathcal{B}(y)) \;=\; \{\text{multi-sets } m=\{b_1,\ldots,b_k\} \text{ with } b_i\in\mathcal{B}\}.$$

This has $|m|=k$ marked by x^k and $\mathrm{wt}(m)=\sum_{i=1}^k|b_k|=n$ marked by y^n:

$$M(x,y)=\sum_{k,n\geq 0}M_k^{(n)}x^ky^n,\quad M_k^{(n)}=\#\{\text{multisets of } k \text{ elements from } \mathcal{B} \text{ with total weight } n\}.$$

Now, the Multiplicity Transform realizes m as a multiplicity function $m:\mathcal{B}\to\mathbb{N}$ with $|m|=\sum_{b\in\mathcal{B}}m(b_i)$ and $\mathrm{wt}(m)=\sum_{b\in\mathcal{B}}m(b_i)|b_i|$. Thus for $\mathbb{N}(x)=\{0,1x,2x^2,\ldots\}$, we get:

$$\mathcal{M}(x,y)\cong\mathbb{N}(x)^{\mathcal{B}(y)},\qquad M(x,y)=\prod_{n\geq 1}(1-xy^n)^{-B_n}.$$

A second formula comes from realizing $\mathrm{MS{\scriptsize ET}}_k\,\mathcal{B}$ as the quotient $\mathcal{B}^k/\mathfrak{S}_k$ and applying Burnside theory. Consider the graded class:*

$$\widetilde{\mathcal{M}} \;=\; \coprod_{k\geq 0}\mathcal{B}(y)^k\tfrac{x^k}{k!} \;\cong\; \{f:[k]\to\mathcal{B} \text{ for } k\in\mathbb{N}\},$$

with $|f|=k$ marked by $\frac{x^k}{k!}$ and $\mathrm{wt}(f)=\sum_{i=1}^k|f(i)|=n$ marked by y^n. Then we have:

$$\mathcal{M} \;=\; \coprod_{k\geq 0}\widetilde{\mathcal{M}}_k/\mathfrak{S}_k.$$

As in the proof of Burnside's Theorem, we consider the total stabilizer class:

$$\coprod_{k\geq 0}\coprod_{\pi\in\mathfrak{S}_k}\widetilde{\mathcal{M}}_k^{\pi}(x,y) \;\cong\; \{(\pi,f)\in\mathfrak{S}_k\times\widetilde{\mathcal{M}}_k \mid k\geq 0, \pi(f)=f\} \;\stackrel{\text{def}}{=}\; \tilde{\mathcal{S}}(x,y),$$

and we apply the Theorem to compute the generating function of the quotient:

$$M(x,y) \;=\; \sum_{k\geq 0}x^k\left(\tfrac{1}{k!}\sum_{\pi\in\mathfrak{S}_k}\tilde{M}_k^{\pi}(y)\right) \;=\; \tilde{S}(x,y).$$

Note that the right side is the *exponential* generating function of $\tilde{\mathcal{S}}$, whereas the left side is the *ordinary* generating function of $\mathcal{M}$: the factor $\frac{1}{k!}$ appears as $\frac{1}{\#G}=\frac{1}{\#\mathfrak{S}_k}$.

*Note that for $\tilde{\mathbb{N}}(x)=\{\varnothing,[1]\frac{x}{1!},[2]\frac{x^2}{2!},\ldots\}$, we have $\widetilde{\mathcal{M}}\cong\ \mathrm{AF{\scriptsize UN}}(\tilde{\mathbb{N}}(x),\mathcal{B}(y))$, so $\tilde{M}(x,y)=\exp(xB(y))$.

Thus we need to compute $\tilde{S}(x,y)$. Now, $\tilde{\mathcal{S}}$ contains those (π, f) for which $f:[k]\to\mathcal{B}$ is constant on each cycle of π. We can construct $\tilde{\mathcal{S}}$ by taking labeled cycles of length $\ell \geq 1$, each tagged with a repeated element of $\Delta^\ell\mathcal{B} = \{(b,\dots,b) \text{ for } b\in\mathcal{B}\}$; then taking sets of these tagged cycles, relabeling the indices to get a permutation π and a function f constant on each cycle:

$$\tilde{\mathcal{S}} \;\cong\; \mathrm{SET}^{\sim} \textstyle\coprod_{\ell\geq 1}\big(\,\mathrm{CYC}_\ell^{\sim}([1]x)\times \Delta^\ell\mathcal{B}(y)\,\big)\,.$$

We deduce our second formula for the generating function of $\mathcal{M}(x,y) = \mathrm{MSET}(x\mathcal{B}(y))$:

$$M(x,y) \;=\; \tilde{S}(x,y) \;=\; \exp\!\left(x\mathcal{B}(y) + \frac{x^2}{2}\mathcal{B}(y^2) + \frac{x^3}{3}\mathcal{B}(y^3) + \cdots\right).$$

Subsets of a class. We can apply the same analysis to $\mathcal{L}(x,y) = \mathrm{SET}(x\mathcal{B}(y))$, the unlabeled class of subsets $s=\{b_1,\dots,b_k\}$ of $\mathcal{B}$-elements with no repeats, with the number of elements $|s|=k$ marked by x^k, and with total weight $\mathrm{wt}(s) = \sum_{i=1}^k |b_k| = n$ marked by y^n. Again, the Multiplicity Transform gives the first formula:

$$\mathcal{L}(x,y)\cong\{0,1x\}^{\mathcal{B}(y)}, \qquad L(x,y) \;=\; \textstyle\sum_{k,n\geq 0} L_k^{(n)} x^k y^n \;=\; \prod_{n\geq 1}(1+xy^n)^{B_n}.$$

To obtain a second formula by considering $\mathcal{L} = \mathrm{SET}\,\mathcal{B}$ as a quotient, we again consider $\widetilde{\mathcal{M}}(x,y) = \{f:[k]\to\mathcal{B} \text{ for } k\geq 0\}$, so that $\mathrm{MSET}_k(\mathcal{B}) \cong \widetilde{\mathcal{M}}_k/\mathfrak{S}_k$, and the total stabilizer:

$$\tilde{\mathcal{S}}(x,y) \;=\; \{(\pi,f)\in\mathfrak{S}_k\times\widetilde{\mathcal{M}}_k \mid k\geq 0, \pi(f)=f\}, \qquad M(x,y) \;=\; \tilde{S}(x,y).$$

Now let $\widetilde{\mathcal{L}} = \{\text{injective } f:[k]\hookrightarrow\mathcal{B} \text{ for } k\geq 0\}$, so that $\mathcal{L}_k \cong \widetilde{\mathcal{L}}_k/\mathfrak{S}_k$. Since $f(1),\dots,f(k)$ are all distinct, $\mathfrak{S}_k$ acts freely on $\widetilde{\mathcal{L}}_k$, i.e. $\pi(f)=f$ only for $\pi=\mathrm{id}$, and $L_k(y) = \widetilde{L}_k(y)/k!$.

$$\widetilde{\mathcal{L}} \;\cong\; \{\mathrm{id}\}\times\widetilde{\mathcal{L}} \;=\; \{(\pi,f)\in\mathfrak{S}_k\times\widetilde{\mathcal{L}} \mid k\geq 0,\ \pi(f)=f\}.$$

We will define a sign function on $\tilde{\mathcal{S}}$, as well as an involution I which cancels non-injective f, so that $\tilde{\mathcal{S}}^I = \{\mathrm{id}\}\times\widetilde{\mathcal{L}}$. Then by the Involution Principle, the signed generating function of $\tilde{\mathcal{S}}$ equals the (positive) signed generating function of the fixed class $\tilde{\mathcal{S}}^I\subset\tilde{\mathcal{S}}^+$:

$$\tilde{S}^{\pm}(x,y) \;=\; \tilde{S}^I(x,y) \;=\; \widetilde{L}(x,y) \;=\; \sum_{k\geq 0}\frac{\widetilde{L}_k(y)}{k!}x^k \;=\; \sum_{k\geq 0} L_k(y)x^k \;=\; L(x,y).$$

That is, the *ordinary* generating function of $\mathcal{L}$ is equal to the *signed exponential* generating function of $\tilde{\mathcal{S}}$. To compute $\tilde{S}^{\pm}(x,y)$, we repeat the multiset construction, but with signs:

$$\tilde{\mathcal{S}} \;\cong\; \mathrm{SET}^{\sim} \textstyle\coprod_{\ell\geq 1}\big(\,(-1)\mathrm{CYC}_\ell^{\sim}((-1)[1]x)\times \Delta^\ell\mathcal{B}(y)\,\big)\,,$$

The signed generating function gives our second formula to count $\mathcal{L}(x,y) = \mathrm{SET}(x\mathcal{B}(y))$:

$$L(x,y) \;=\; \tilde{S}^{\pm}(x,y) \;=\; \exp\!\left(x\mathcal{B}(y) - \frac{x^2}{2}\mathcal{B}(y^2) + \frac{x^3}{3}\mathcal{B}(y^3) - \cdots\right).$$

Lastly, we define the promised sign function and involution on $(\pi,f)\in\tilde{\mathcal{S}}_k$. Let $\mathrm{sgn}(\pi,f) = (-1)^{k-\mathrm{cyc}(\pi)}$. For $(\mathrm{id},f)\in\{\mathrm{id}\}\times\widetilde{\mathcal{L}}$ with f injective, let $I(\mathrm{id},f) = (\mathrm{id},f)$. Otherwise, if $(\pi,f)\in\tilde{\mathcal{S}}^{\pm}$ with $f:[k]\to\mathcal{B}$ not injective, suppose $f(i)=f(j)$ for minimal

$i,j \in [k]$; then define $I(\pi, f) = (\pi \cdot (ij), f)$, multiplying π by the transposition (ij), so that $\pi \cdot (ij) = (\ell m) \cdot \pi$ for $\ell = \pi(a)$, $m = \pi(j)$. This reverses the sign $(-1)^{k-\mathrm{cyc}(\pi)}$, since if i,j lie in the same cycle of π, then $\pi \cdot (ij)$ cuts this cycle into two; if i,j lie on different cycles, then $\pi \cdot (ij)$ joins these cycles into one. Thus, I pairs off all non-injective $(\pi, f) \in \tilde{\mathcal{S}}^{\pm}$, leaving only the injective (id, f).

We call I the 0/00 Involution because it splits a single cycle 0 into two cycles 00, and vice versa. It is useful because it toggles both obvious definitions of sign for π, incrementing/decrementing both the number of cycles and the number of transpositions.

Poset constructions. The ideas of Doubilet-Rota-Stanley give a direct connection between combinatorial structures and generating series: semi-infinite ranked posets give algebra structures to graded classes, and generating series emerge as subalgebras.

A *poset* is a class $\mathcal{P}$ with a partial order relation $a < b$ which is anti-symmetric ($a < b \Rightarrow b \not< a$) and transitive ($a<b<c \Rightarrow a<c$), and we define $a \leq b$ to mean $a < b$ or $a = b$. A *covering* $a \lessdot b$ means $a < b$ with no intermediate elements $a < c < b$. If $\mathcal{P}$ has a unique minimal element, we denote it as $\hat{0} \leq a$ for all $a \in \mathcal{P}$. and similarly for a unique maximal element $\hat{1} \geq a$.

An interval is a sub-poset $[a,b] = \{c \in \mathcal{P} \text{ with } a \leq c \leq b\}$. A *chain* of length ℓ from a to b is an increasing sequence $a = a_0 < a_1 < \cdots < a_\ell = b$, and it is a *saturated chain* if each inequality is a covering. A *ranked poset* is a graded class $\mathcal{P} = \coprod_{k \geq 0} \mathcal{P}_k$ with size function $|a| = \mathrm{rk}(a)$ such that any covering $a \lessdot b$ has $\mathrm{rk}(a) + 1 = \mathrm{rk}(b)$; the length of an interval $[a,b]$ is defined as $\ell(a,b) = \mathrm{rk}(b) - \mathrm{rk}(a)$.† An *antichain* of length ℓ is a set of elements $\{a_1, \ldots, a_\ell\}$ with no inequalities among them, $a_i \not< a_j$.

The *incidence algebra* of a poset is defined as:

$$I(\mathcal{P}) \;=\; \{\alpha : \mathrm{Int}(\mathcal{P}) \to \mathbb{C}\} \;\cong\; \textstyle\bigoplus_{a \leq b} \mathbb{C}\,[a,b],$$

all functions on the set of intervals $\mathrm{Int}(\mathcal{P}) = \{[a,b] \text{ for } a \leq b\}$; a function α can be written as a formal linear combination of intervals: $\alpha = \sum_{a \leq b} \alpha(a,b)\,[a,b]$. Functions are multiplied by convolution, which is equivalent to concatenation of intervals:

$$(\alpha \cdot \beta)(a,b) = \sum_{a \leq c \leq b} \alpha(a,c)\,\beta(c,b), \qquad [a,b] \cdot [c,d] = \begin{cases} [a,d] & \text{if } b = c \\ 0 & \text{otherwise.} \end{cases}$$

Any *linear extension* $e : \mathcal{P} \to \mathbb{Z}$, meaning $a < b \Rightarrow e(a) < e(b)$, induces an injective homomorphism from $I(\mathcal{P})$ to the algebra of upper-triangular matrices, with the basis element $[a,b] \in I(\mathcal{P})$ mapping to the coordinate matrix $E_{e(a),e(b)}$ in $M_{n \times n}(\mathbb{C})$ if $n = \#\mathcal{P} < \infty$, or in $M_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{C})$ if $\mathcal{P}$ is countably infinite.

The identity element of $I(\mathcal{P})$ is the *delta function* $\delta(a,a) = 1$ and $\delta(a,b) = 0$ for $a < b$. An element α has a reciprocal $\alpha^{-1} \in I(\mathcal{P})$ whenever $\alpha(a,a) \neq 0$ for all a. The *zeta-function* is defined as $\zeta(a,b) = 1$ for all $a \leq b$, and its reciprocal is the *Möbius function* $\mu = \zeta^{-1}$. Then $\mu \cdot \zeta = \delta$ is equivalent to $\mu(a,a) = 1$ and $\sum_{c \in [a,b]} \mu(a,c) = 0$ for $a < b$, and to the recursive formula $\mu(a,b) = -\sum_{a \leq c < b} \mu(a,c)$.

Mobius Inversion Formula: For functions $f, g : \mathcal{P} \to \mathbb{C}$, we have two pairs of equivalences:

$$f(a) = \sum_{b \geq a} g(b) \iff g(a) = \sum_{b \geq a} \mu(a,b) g(b), \quad \text{and} \quad f(b) = \sum_{a \leq b} g(a) \iff g(b) = \sum_{a \leq b} g(a)\,\mu(a,b).$$

† Such $\mathrm{rk}(a)$ exists (essentially uniquely) if every saturated chain from a to b has the same length $\ell(a,b)$.

Proof: Let $I(\mathcal{P})$ act on the vector space of functions $\mathbb{C}[\mathcal{P}] = \{f : \mathcal{P} \to \mathbb{C}\}$ as a left module via $(\alpha \cdot f)(a) = \sum_{b \geq a} \alpha(a,b) f(b)$, so that $\alpha \cdot (\beta \cdot f) = (\alpha \cdot \beta) \cdot f$.[‡] Then we have:

$$f(a) = \sum_{b \geq a} g(b) \iff f = \zeta \cdot g \iff \mu \cdot f = \mu \cdot \zeta \cdot g = g \iff g(a) = \sum_{b \geq a} \mu(a,b) f(b).$$

The other equivalence follows from the right action $(f \cdot \alpha)(b) = \sum_{a \leq b} f(a)\, \alpha(a,b)$. $\square$

A *binomial poset* is a ranked poset with a distinguished infinite chain $\hat{0} = \tilde{0} \lessdot \tilde{1} \lessdot \tilde{2} \lessdot \cdots$, such that for every interval $[a,b]$ with length $\ell(a,b) = n$, the set $\mathcal{B}(a,b)$ of saturated chains from a to b has the same number of elements, $\#\mathcal{B}(a,b) = \#\mathcal{B}(\tilde{0},\tilde{n}) = B(n)$, where $B(0) = B(1) = 1$. In $I(\mathcal{P})$, define the elements:

$$\bar{n} = \textstyle\sum_{\ell(a,b)=n} [a,b], \qquad x = \bar{1} = \sum_{a \lessdot b} [a,b].$$

Then:

$$x^n \;=\; \textstyle\sum_{(a \lessdot \cdots \lessdot b) \in \mathcal{B}(a,b)} [a,b] \;=\; B(n)\, \bar{n}.$$

Now define the *reduced incidence algebra*:

$$R(\mathcal{P}) \;=\; \bigoplus_{n \geq 0} \mathbb{C}\, \tfrac{x^n}{B(n)} \;=\; \bigoplus_{n \geq 0} \mathbb{C}\, \overline{n} \;=\; \{\alpha \in I(\mathcal{P}) \text{ with } \alpha(a,b) = \alpha(\tilde{0},\tilde{n}) \text{ for } n = \ell(a,b)\}.$$

We can write elements of $R(\mathcal{P})$ as power series:

$$\alpha = a_0 + a_1 x + a_2 \tfrac{x^2}{B(2)} + a_3 \tfrac{x^3}{B(3)} + \cdots,$$

where the term a_0 means $a_0 \delta$. That is, $R(\mathcal{P})$ is isomorphic to the formal power series ring $\mathbb{C}[[x]]$ with basis $x^n / B(n)$, and we may transfer the x-adic topology to $R(\mathcal{P})$. The zeta and Mobius functions lie in $R(\mathcal{P})$:

$$\zeta \;=\; \sum_{n \geq 0} \frac{x^n}{B(n)}, \qquad \mu \;=\; \frac{1}{\zeta} \;=\; \frac{1}{1 + (\zeta - 1)} \;=\; 1 - (\zeta - 1) + (\zeta - 1)^2 - \cdots.$$

This series converges because the n^{th} term contains only components $[a,b]$ with $\ell(a,b) \geq n$.

‡This is isomorphic to the natural action of the embedding of $I(\mathcal{P})$ into a matrix algebra.

************************************ What is the real meaning of PIE?
(i) Mobius inversion for functions on the binomial poset $\mathcal{P} = \wp[n]$.
(ii) Involution principle, cancellation in a signed class $\wp[n] \times \mathcal{A}$.
(iii) Algebra of characteristic functions in $\mathbb{C}[\mathcal{A}]$.
(iv) Some kind of universal framework in $I(\mathcal{P})$ including the other approaches

****
Is there a Taylor's coefficient formula for general $R(\mathcal{P})$, generalizing calc of finite differences for $\mathcal{P} = \mathbb{N}$ with $\mu \cdot f = \Delta f$?

****
Examples: chain, Boolean (product), divisor poset (product, non-binomial; necklace poly Moreau $M_n(k)$)

Stratification posets. Geometric lattices, hyperplane arrangements, set partitions, Mobius via characteristic polynomial counting. Regular cell complexes, simpicial complexes and Hall's Theorem, $\Delta(\mathcal{P}(\Delta))$ is barycentric subdivision. Is every important poset a stratification poset?

Product posets: For two posets $\mathcal{P}, \mathcal{Q}$, their Cartesian product $\mathcal{P} \times \mathcal{Q}$ is defined by $(p, q) \le (p', q')$ whenever $p \le p'$ and $q \le q'$. The Mobius function of a product poset is the product of the Mobius functions of the factors: $\mu_{\mathcal{P}\times\mathcal{Q}}((p, q), (p', q')) = \mu_{\mathcal{P}}(p, p')\mu_{\mathcal{Q}}(q, q')$, as is easily checked using the recursive formula above.

Example: Divisor poset $\mathcal{D} = \{1, 2, 3, \ldots\}$ ordered by integer divisibility written as $a|b$ instead of $a \le b$, and minimal element 1 instead of $\hat{0}$. This is not a binomial poset, but has similar features. Its standard elements $\tilde{n} = n$ do not form a chain, but every interval is isomorphic to a standard interval: $[a, b] \cong [1, b/a]$.[§] Define the reduced incidence algebra:

$$R(\mathcal{P}) \;=\; \bigoplus_{n\ge 0} \mathbb{C}\,\overline{n} \;=\; \{\alpha \in I(\mathcal{P}) \text{ with } \alpha(a, b) = \alpha(1, n) \text{ for } n = b/a\}.$$

We can easily check the character law $\bar{n} \cdot \bar{m} = \overline{nm}$, which is the same as the multiplication of the functions n^s for a complex variable s. Thus $R(P)$ is isomorphic to the algebra of Dirichlet series $\bigoplus_{n\ge 0} \mathbb{C}n^s$ via $\bar{n} \mapsto n^s$, and we may identify $\alpha \in R(\mathcal{P})$ with the function

$$\alpha(s) = \tfrac{a_1}{1^s} + \tfrac{a_2}{2^s} + \tfrac{a_3}{3^s} + \cdots \qquad \text{for} \qquad a_n = \alpha(1, n).$$

Again, $R(\mathcal{P})$ contains $\delta(s) = 1.$ and $\zeta(s)$ is the classical Riemann zeta function. Its reciprocal $\mu(s) = 1/\zeta(s) = \sum_{n\ge 0} \mu(n)/n^s$ is the original case for which Mobius introduced his function.
****

Example: q-Boolean subspaces, q-Binomial Theorem in $R(\mathcal{P})$.

Use q-commuting twisted $R(\mathcal{P})_q$, expanded to a flag algebra, and twisted with vector-markings. Prove $yx = qxy \Rightarrow (x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}$, via coordinate-free Schubert cell mapping. Quantum function algebras.

[§] $[a, b] \cong [c, d]$ whenever b/a and c/d have the same number of prime factors with the same multiplicites.

Koornwinder: How this relates to $(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^{n} q^{\binom{k}{2}} x^k/[k]_q^!$. Mention q-analog of $(1+\frac{x}{n})^n \to e^x$.

Also Macdonald-book type formulas like $\prod_{i=1}^{n}(1+t^k x_i) = \sum_{k=0}^{n} t^k e_i(x_1,\ldots,x_n)$ Cauchy identity & generalization to dual bases in Λ

****

Two "dual" algebras via Britz-Fomin, Dilworth-Greene chain-antichain duality.

1. Order ideal lattice $J(\mathcal{P})$ with antichain basis

2. Flag algebra $\mathbb{C}\Delta[\mathcal{P}]$ with chain basis, i.e. the chain complex of simplicial complex $\Delta(\mathcal{P})$, with concatenation multiplication = cup product. Beilinson-Kazhdan-Macpherson on quantum function algebras. Or is it quantum enveloping algebra?

****

Summary of asymptotic theorems & examples from green spiral notebook: poles, comparison theorems, saddle point for Stirling.

****

PROBLEMS

1. The formula $\zeta\cdot\mu=\delta$ is equivalent to the equations: $\mu[a,a]=1$ and $\sum_{a\le c\le b}\mu[a,c]=0$ for $a<b$. The direct product of two posets is $\mathcal{P}\times\mathcal{Q}$, with $(p,q)\le(p',q')$ whenever $p\le p'$ and $q\le q'$. Prove that the Möbius function of the product is the product of the individual Möbius functions of $\mathcal{P},\mathcal{Q}$:

$$\mu_{\mathcal{P}\times\mathcal{Q}}[(p,q),(p',q')] \;=\; \mu_{\mathcal{P}}[p,p']\;\mu_{\mathcal{Q}}[q,q'].$$

2. For the poset $\mathcal{P}=\mathcal{D}_{18}$, the 6-element poset of divisors of $18=2\cdot 3^2$ ordered by divisibility, work out the Möbius function $\mu[a,b]$ in several ways:

a. Write a 6×6 matrix Z corresponding to $\zeta[a,b]=1$ for all $a\le b$, and invert by Gaussian elimination: write a double matrix $[Z\,|\,I]$, then row reduce to the form $[\,I\,|\,M]$, so that $M=Z^{-1}$.

b. Write $Z=I+N$, for identity I and strictly upper-triangular N, nilpotent with $N^6=0$. By computer, expand the geometric series $M=(I+N)^{-1}=I-N+N^2-\cdots$.

c. For each $a\in\mathcal{P}$, draw a copy of the Hasse diagram (a 2×3 rectangle). Mark $\mu[a,a]=1$, then work upwards computing $\mu[a,b]$ using the recurrence $\mu[a,b]=-\sum_{a\le c<b}\mu[a,c]$.

d. Apply the product formula of #1 above to $\mathcal{D}_{18}\cong[2]\times[3]$, the direct product of two chains. Match this with Mobius' original description: $\mu[d,n]=(-1)^k$ if n/d is the product of k distinct primes, and $\mu[d,n]=0$ if n/d is divisible by a square number.

e. Evaluate Phillip Hall's Formula: $\mu[a,b]=\hat{c}_0-\hat{c}_1+\hat{c}_2-\cdots$, where $\hat{c}_d$ is the number of chains of length d from a to b in $\mathcal{P}$, starting with $\hat{c}_0=0,\hat{c}_1=1$.

f. Consider $\mathcal{P}=\mathcal{Q}\sqcup\{\hat{0},\hat{1}\}$, where $\mathcal{Q}=\{a\in\mathcal{P}$ with $\hat{0}<a<\hat{1}\}$, and form the simplicial complex $\Delta(Q)$ whose elements are all chains in $\mathcal{Q}$. Draw a picture of the corresponding topological space, consisting of one-simplexes glued at their endpoints.

Hall's Formula says $\mu[\hat{0},\hat{1}] = \tilde{\chi}(\Delta(Q))$, the reduced Euler characteristic of the above topological space, the alternating sum of the number of simplexes of each dimension, minus 1. Compute $\tilde{\chi}(\Delta(Q))$ from this definition. Also, find the simplest triangulation of this space, and compute $\tilde{\chi}$ from that.

3. The posets $\mathcal{D}_n$ of divisors of n have the semi-infinite union $\mathcal{P} = \mathcal{D}_\infty = \{1,2,3,\ldots\}$ ordered by divisibility. This has standard elements $\hat{n} = n$, and the equivalence of intervals $[a,b] \sim [c,d]$ whenever $b/a = d/c$,[¶] which induces equivalence classes $\bar{n} = \overline{[1,n]}$, making a basis of the reduced algebra $R(\mathcal{P}) = \bigoplus_{n\geq 1} \mathbb{C}\,\bar{n}$. We have $\bar{n}\bar{m} = \overline{nm}$, so $R(\mathcal{P})$ embeds in the ring of complex functions via $\bar{n} \cong n^{-s}$, and $\alpha \in R(\mathcal{P})$ corresponds to a Dirichlet series $\sum_{n\geq 1} \frac{\alpha(\bar{n})}{n^s}$, where s is a complex variable.

a. Recall how we counted necklaces of n beads chosen from k colors, orbits of the cyclic symmetry group $G = C_n$. Since G has $\phi(n/d)$ permutations with d cycles, Burnside's Theorem showed that the number of orbits is the necklace polynomial:

$$N_n(k) = \tfrac{1}{\#G} \textstyle\sum_{\pi\in G} k^{\mathrm{cyc}(\pi)} = \tfrac{1}{n} \sum_{d|n} \phi(n/d)\, k^d.$$

Problem: Count the number $M_n(k)$ of *aperiodic* necklaces, those with no cyclic symmetry, so their orbit has size n. We use a form of inclusion-exclusion. Show that:

$$k^n = \textstyle\sum_{d|n} dM_d(k).$$

That is, if we consider $\alpha(n) = k^n$ and $\beta(n) = nM_n(k)$ as elements of $R(\mathcal{P})$, we have $\alpha = \zeta \cdot \beta$. Now give a formula for $M_n(k)$ via Mobius inversion.

Note: For an extension of finite fields $\mathbb{F}_p \subset \mathbb{F}_q$ with $q = p^n$, the cyclic group $G = C_n$ is the Galois group, generated by the Frobenius automorphism $\Phi(s) = s^p$. Elements of $\mathbb{F}_{p^n}$ can be written as $a_1 s_\circ + a_2\Phi(s_\circ) + \cdots + a_n\Phi^{n-1}(s_\circ)$ for a fixed $s_\circ \in \mathbb{F}_q$ and arbitrary $a_i \in \mathbb{F}_p$, so we may consider the coefficients as taking the role of $k = p$ colors in a necklace. An orbit of G comprises the roots of an irreducible polynomial over $\mathbb{F}_p$, and orbits of size n are the roots of irreducible polynomials of degree n. Thus $M_n(p)$ counts monic irreducible polynomials of degree n in $\mathbb{F}_p[x]$.

b. $\mathcal{D}_n$ is a lattice. What is the usual number-theory interpretation of the meet $a \vee b$ and the join $a \wedge b$?

[*Added*] **c.** Show that $\mathcal{D}_n$ is distributive. Find its join-irreducible elements, and show that $k = \bigvee j$ where j runs over all join-irreducibles $\leq k$.

[¶] This is stronger than rank equivalence $\ell[a,b] = \ell[c,d]$, and isomorphism equivalence $[a,b] \cong [c,d]$.

4. For a finite field $F = \mathbb{F}_q$, consider the poset $\mathcal{B}_n(q)$ of linear subspaces $V \subset F^n$ ordered by inclusion, a q-analog of the Boolean poset $\mathcal{B}_n \cong \wp[n]$ of subsets $I \subset [n]$. The union of these spaces via the inclusions $0 \subset F^1 \subset F^2 \subset \cdots$ is the semi-infinte poset $\mathcal{P} = \mathcal{B}_\infty(q)$, with standard elements $\hat{n} = F^n$. The reduced incidence algebra $R(\mathcal{P})$ is isomorphic to $\mathbb{C}[[x]]$, with the basis element $\bar{n}$ corresponding to $x^n/[n]_q^!$. Thus we can consider $R(\mathcal{P})$ as the ring of Eulerian generating functions:

$$f(x) \;=\; \sum_{n\geq 0} a_n \frac{x^n}{[n]_q^!} \;=\; a_0 + a_1 x + a_2 \frac{x^2}{1{+}q} + a_3 \frac{x^3}{(1{+}q{+}q^2)(1{+}q)} + \cdots .$$

The zeta function of $\mathcal{P}$ is $\zeta = \exp_q(x) = \sum_{n\geq 0} \frac{x^n}{[n]_q^!}$.

a. Explain why $\mathcal{P}$ is a binomial poset with

$$B(n) = [n]_q^! = \#\mathrm{Flag}(\mathbb{F}_q^n) = [n]_q[n{-}1]_q \cdots [2]_q[1]_q, \quad \text{where } [n]_q = \tfrac{q^n-1}{q-1}.$$

b. Show that the reciprocal of $\zeta = \exp_q(x)$ is the powersd series

$$\mu \;=\; \sum_{n\geq 0} (-1)^n q^{\binom{n}{2}} \frac{x^n}{[n]_q^!},$$

and determine the Mobius function $\mu[U,V]$ for any $U \subset V$ in $\mathcal{P}$.
Hint: Use the q-binomial theorem $\prod_{i=1}^n (1 + q^{i-1}x) = \sum_{k=0}^n q^{\binom{k}{2}} \binom{n}{k}_q x^k$.

c. For an n-dimensional $V \in \mathcal{P}$, let

$$\alpha(V) \;=\; \alpha(n) \;=\; \#\{\text{linear functions}\, f : F^k \to V\} \;=\; q^{nk},$$
$$\beta(V) \;=\; \beta(n) \;=\; \#\{\text{surjective}\, f : F^k \to V\} \;=\; \mathrm{surj}_q(k,n).$$

Every $f : F^k \to F^n$ is surjective onto its image $V = f(F^k)$, so that

$$\alpha(n) \;=\; \sum_{V \subset F^n} \beta(V).$$

Solve for $\beta(n)$ by Mobius inversion, obtaining an explicit summation formula for the number of surjective linear mappings $f : F^k \to F^n$.

d. Show that the number of surjective linear mappings $f : F^k \to F^n$ is equal to the number of injective linear mappings $f : F^n \to F^k$. Determine this last number directly, obtaining a product formula much simpler than in part (c). Verify algebraically that these formulas are equal for $n = 1$.

NOTES: The Grassmannian $\mathrm{Gr}(d, F^n)$ is the parameter space whose points correspond to d-dimensional subspaces V in the n-dimensional vector space F^n over a given field F. We specify a subspace $V = \mathrm{Span}_F(v_1, \ldots, v_d)$ by a $d{\times}n$ matrix of row vectors, with change-of-basis symmetry group $\mathrm{GL}_d(F)$. This matrix can be normalized by making a given $d \times d$ submatrix into the identity, in columns $I = \{i_1 < \cdots < i_d\} \subset [n]$, provided the determinant of this submatrix is nonzero:

$$V \;=\; \mathrm{GL}_d \circlearrowleft \begin{bmatrix} \text{——} \; v_1 \; \text{——} \\ \text{——} \; v_2 \; \text{——} \\ \vdots \\ \text{——} \; v_d \; \text{——} \end{bmatrix} \;=\; \begin{bmatrix} * * \cdots 1 \cdots 0 \cdots 0 \cdots * * \\ * * \cdots 0 \cdots 1 \cdots 0 \cdots * * \\ \vdots \\ * * \cdots 0 \cdots 0 \cdots 1 \cdots * * \end{bmatrix}$$

The $*$'s denote $d(n{-}d)$ free parameters in F defining a coordinate chart U_I of the Grassmannian, making it into an F-manifold: $\mathrm{Gr}(d,F^n) \;=\; \bigcup_I U_I$.

We define the Schubert cell decomposition $\mathrm{Gr}(d,F^n) \;=\; \coprod_I X_I$ by letting X_I consist of those $V \in U_I$ which have no $*$'s to the right of any 1 (row-echelon form). We can define X_I geometrically in terms of the standard basis $\{e_1,\ldots,e_n\}$ of F^n and the standard coordinate subspaces $E_r = \mathrm{Span}(e_1,\ldots,e_r)$; then $V \in X_I$ whenever $\dim(V{\cap}E_r) = \#(I{\cap}[r])$ for $r = 1,\ldots,n$. That is, $I = [d]$ forces $V = E_d$, and larger I makes $V \in X_I$ stick out further from the standard subspaces, until $I = \{n{-}d{+}1,\ldots,n\}$ corresponds to generic V's in the open set $X_I = U_I$. The topological closure $\overline{X}_I$ is given by: $\dim(V{\cap}E_r) \geq \#(I{\cap}[r])$ for $r = 1,\ldots,n$. We keep track of how the cells fit together using the Bruhat degeneration order: we define $I \leq J$ to mean $X_I \subset \overline{X}_J$, or equivalently $\overline{X}_I \subset \overline{X}_J$.

Example: For $\mathrm{Gr}(2,F^4)$, we have:

$$U_{34} \;=\; X_{34} \;=\; \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix} \;=\; \{V \mid V{\cap}E_2 = 0\,, \dim(V{\cap}E_3) = 1\},$$

$$U_{14} \;=\; \begin{bmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{bmatrix}, \qquad X_{14} \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix} \;=\; \{V \mid E_1 \subset V \not\subset E_3\}.$$

Here are the cell closures with defining conditions, height indicating Bruhat order:

$$\overline{X}_{34} = \mathrm{Gr}(2,F^4)$$

$$\overline{X}_{24} = (\dim(V{\cap}E_2){\geq}1)$$

$$\overline{X}_{23} = (V{\subset}E_3) \qquad\qquad \overline{X}_{14} = (E_1{\subset}V)$$

$$\overline{X}_{13} = (E_1{\subset}V{\subset}E_3)$$

$$\overline{X}_{12} = (V{=}E_2)$$

The Bruhat order relations $\overline{X}_I \subset \overline{X}_J$ are evident from the defining conditions on V. To verify in coordinates that $\{1,3\} \leq \{1,4\}$, we show that any plane $V_\circ \in X_{13}$ is approached by planes in X_{14}: we find a continuous family $\mathcal{V} : F \to \mathrm{Gr}(2,F^4)$ with $\mathcal{V}(t) \in X_{14}$ for $t \neq 0$, and $\mathcal{V}(0) = V_\circ$:

$$V_\circ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 \end{bmatrix}, \qquad \mathcal{V}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a/t & 1/t & 1 \end{bmatrix} \text{ for } t \neq 0.$$

Similarly, the flag manifold $\mathrm{Fl}(F^n)$ is the parameter space of flags

$$V_\bullet = (0 \subset V_1 \subset \cdots \subset V_{n-1} \subset F^n), \quad \dim(V_d) = d.$$

We specify $V_\bullet$ by a basis $\{v_1,\ldots,v_n\}$ of F^n, with $V_d = \mathrm{Span}(v_1,\ldots,v_d)$; the basis forms an $n{\times}n$ matrix of row vectors. The change-of-basis symmetry group B of $V_\bullet$ consists of all lower-triangular matrices (with non-zero diagonal entries) in $\mathrm{GL}_n(F)$, since we can add a multiple of v_i only to a later basis vector to leave each V_d invariant. We get a Schubert cell decomposition indexed by permutations $w \in S_n$: $\mathrm{Fl}(F^n) = \coprod_w X_w$, where X_w consists of $V_\bullet$ whose B-reduced form is a permutation matrix w, plus $*$ coordinates in the positions of the Röthe diagram $D(w) = \{(i,j) \mid j < w(i),\ i < w^{-1}(j)\}$. Thus $\dim(X_w) = \#D(w) = \mathrm{inv}(w)$.

Problems.

1a. Determine the Gaussian binomial coefficient $\binom{6}{3}_q = \#\mathrm{Gr}(3,\mathbb{F}_q^6)$ as the number of $3{\times}6$ V-basis matrices divided by the number of $3{\times}3$ change-of-basis matrices; also as a quotient of q-integers $[n]_q = \frac{q^n-1}{q-1}$; and finally as a polynomial by dividing through (with computer).

b. There are $\binom{6}{3} = 20$ sets $I = \{i_1, i_2, i_3\} \subset [6]$ indexing the Schubert cells, in bijection with partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0)$ with $\lambda_1 \leq 6{-}3$: the Young diagrams fit in $3{\times}(6{-}3)$. List all I's and λ's, along with the size measure $q^{\mathrm{wt}(I)} = q^{|\lambda|} = \#X_I$. Here $\mathrm{wt}(I) = |\lambda| = \dim(X_I)$. Compare with part (a).

2a. Verify the q-Binomial Theorem:

$$\prod_{i=1}^{n}(1{+}q^i x) \;=\; \sum_{d=0}^{n} q^{\binom{d+1}{2}} \binom{n}{k}_q x^d$$

for the special case $n = 3$. Multiply out by hand!

b. Prove the q-Binomial Theorem for all n by writing the lefthand side as a bivariate generating function for the class of all subsets $I \subset [n]$, then using our expansion illustrated in Prob. 1, $\binom{n}{d}_q = \sum_I q^{\mathrm{wt}(I)}$, where the sum is over all $I = \{i_1 < \cdots < i_d\}$ and $\mathrm{wt}(I) = \sum_{j=1}^{d}(i_j{-}j)$.

3. Find q-analogs of the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ using two analogs of deletion.

a. Consider the mapping $\mathrm{Gr}(d,F^n) \to \mathrm{Gr}(d,F^{n-1}) \coprod \mathrm{Gr}(d{-}1,F^{n-1})$ which takes V to $V' = V \cap E_{n-1}$, intersecting with the coordinate subspace E_{n-1}. Make this a bijection by keeping track of lost information needed to reconstruct the row-echelon basis of V from V'. Deduce a recurrence for $\binom{n}{d}_q$.

b. Let $P : F^n \to F^{n-1}$ be the projection map along e_1, so that $P(e_i) = e_i$ for $i = 2, \ldots, n$. Consider the mapping $\mathrm{Gr}(d,F^n) \to \mathrm{Gr}(d,F^{n-1}) \coprod \mathrm{Gr}(d{-}1,F^{n-1})$ which takes V to $V' = P(V)$. Again make this a bijection, and find a different recurrence for for $\binom{n}{d}_q$.

4a. For each Schubert cell $X_w \subset \mathrm{Fl}(F^3)$ corresponding to $w \in S_3$, write the B-reduced matrix form for $V_\bullet$ corresponding to the Röthe diagram $D(w)$. For $F = \mathbb{F}_q$, explicitly verify:

$$\#\mathrm{Flag}(\mathbb{F}_q^3) \;=\; \frac{\#\mathrm{GL}_3(\mathbb{F}_q)}{\#B} \;=\; [3]_q[2]_q[1]_q \;=\; \sum_{w\in S_3} q^{\mathrm{inv}(w)}.$$

b. For each Schubert cell closure $\overline{X}_w$ above, describe its flags $V_\bullet = (V_1 \subset V_2)$ in terms of their relation to the standard flag $E_1 \subset E_2$. For example, the minimal cell closure is: $\overline{X}_{123} = X_{123} = \{E_\bullet\} = \{V_\bullet \mid V_1 {=} E_1\,,\, V_2 {=} E_2\}$. By examining the implications among the defining conditions for the $\overline{X}_w$'s, arrange the w's according their Bruhat degeneration order, defined by $\overline{X}_w \subset \overline{X}_u$.

c. The Bruhat order covering relations $w' {\lessdot} w$ correspond to minimal containments $\overline{X}_{w'} \subset \overline{X}_w$, having $\dim X_{w'} = \dim X_w - 1$. For each minimal containment in $\mathrm{Flag}(F^3)$ and any $(V_\bullet) \in X_{w'}$ (properly in the cell, not its closure), give a family $\mathcal{V} : F \to \mathrm{Fl}(F^3)$ with $\mathcal{V}(t) \in X_w$ for $t \neq 0$ and $\mathcal{V}(0) = V_\bullet$. Also, describe each covering combinatorially in terms of moving a certain pair of 1's in the permutation matrix of w to get w', and conjecture a general move rule for coverings of $w \in S_n$.