**Bigraded classes.** Following Flajolet-Sedgewick Ch. III, we define a *bigraded class* $\mathcal{A}$ to be a set of combinatorial objects $a \in \mathcal{A}$ with *two* measures of magnitude, a primary measure $|a| = n$ called simply the "size", and a secondary measure $||a|| = k$ called the "weight" or "parameter", or a "statistic on $\mathcal{A}$". Usually we consider *labeled* bigraded classes $\tilde{\mathcal{A}}$, in which each object $a \in \tilde{\mathcal{A}}$ has $n = |a|$ atoms having all the labels $1, 2, \ldots, n$, as well as the weight $||a||$ unrelated to the labels: in particular, a permutation of the labels should not change either $|a|$ or $||a||$. We have the counting numbers $A_n^{(k)} = \#\{a \in \tilde{\mathcal{A}} \text{ with } |a| = n,\, ||a|| = k\}$.

The classic example is the labeled class $\tilde{\mathcal{P}} = \coprod_{n\geq 0} \mathfrak{S}_n$ comprising all permutations $w \in \mathfrak{S}_n$, with size $|w| = n$ and weight $||w|| = \mathrm{cyc}(n) =$ number of cycles of $w$. In this case the counting numbers are the *Stirling cycle numbers*: $\left[{n \atop k}\right] = P_n^{(k)} = \#\{w \in S_n \mid \mathrm{cyc}(w) = k\}$.

We take the *bivariate generating function*:

$$\tilde{A}(x,t) \;=\; \sum_{n,k\geq 0} A_n^{(k)} \frac{x^n}{n!}\, t^k \;=\; \sum_{a\in\tilde{\mathcal{A}}} \frac{x^{|a|}}{|a|!}\, t^{||a||}\,.$$

All the constructions available for labeled graded classes extend to the bigraded case. In particular, we endow the labeled product $\tilde{\mathcal{A}} * \tilde{\mathcal{B}}$ with the additive weight function $||(a_S, b_T)|| = ||a|| + ||b||$, where $a_S$ means $a \in \tilde{\mathcal{A}}$ with its atoms relabeled by the set $S$.

We have the Labeled Bigraded Product Priniciple: for $\tilde{\mathcal{C}} = \tilde{\mathcal{A}} * \tilde{\mathcal{B}}$, the bivariate generating function is $\tilde{C}(x,t) = \tilde{A}(x,t)\cdot\tilde{B}(x,t)$. The proof is very similar to the single-graded case:

$$\begin{aligned}
\tilde{A}(x,t)\cdot\tilde{B}(x,t) &= \Big(\sum_{p\geq 0}\sum_{i\geq 0} A_p^{(i)}\frac{x^p}{p!}\,t^i\Big)\cdot\Big(\sum_{q\geq 0}\sum_{j\geq 0} B_q^{(j)}\frac{x^q}{q!}\,t^j\Big)\\
&= \sum_{n,k\geq 0}\left(\sum_{p=0}^{n}\sum_{i=0}^{k}\binom{n}{p} A_p^{(i)} B_{n-p}^{(k-i)}\right)\frac{x^n}{n!}\,t^k\\
&= \sum_{n,k\geq 0} C_n^{(k)}\frac{x^n}{n!}\,t^k \;=\; \tilde{C}(x,t).
\end{aligned}$$

The second inequality uses the change of index variables $n = p+q$ and $k = i+j$, so that $\binom{n}{p}\frac{1}{n!} = \frac{1}{p!q!}$. The third equality is because each element $(a_S, b_T) \in \tilde{\mathcal{C}}_n^{(k)}$ corresponds to the choice of $S \subset [n]$, $a \in \tilde{\mathcal{A}}_p^{(i)}$, $b \in \tilde{\mathcal{B}}_{n-p}^{(k-i)}$.

Similarly, the constructions $\textsc{Seq}_j$, $\textsc{Seq}$, $\textsc{Set}_j$, $\textsc{Set}$, $\textsc{Cyc}_j$, $\textsc{Cyc}$ can be performed on bigraded classes, and give the same formulas for the bivariate generating functions.

**Counting cycles in permutations.** The class of labeled cycles of length $n$ is $\textsc{Cyc}_n[\widetilde{1}]$, where $[\widetilde{1}]$ is the single-element labeled graded class, with bivariate exponential generating function $\frac{n!}{n}\frac{x^n}{n!}\,t = \frac{x^n}{n}\,t$, since a cycle is the same as a permutation of $[n]$ up to rotation equivalence. Here $\frac{x^n}{n!}$ indicates the size $n$, while $t = t^1$ indicates that a single cycle has weight 1.

Allowing cycles of any length gives generating function $\sum_{n\geq 1}\frac{x^n}{n}\,t = t\log(\frac{1}{1-x})$. Realizing any permutation as a set of $k$ labeled cycles gives the bivariate generating function of

$\tilde{\mathcal{P}} = \coprod_{n\geq 0} S_n = \textsc{Set}(\textsc{Cyc}\widetilde{[1]})$:

$$\tilde{P}(x,t) \;=\; \sum_{n,k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!}\, t^k \;=\; \sum_{k\geq 0} \frac{1}{k!}\Big(t\log(\tfrac{1}{1-x})\Big)^k \;=\; \exp\Big(t\log(\tfrac{1}{1-x})\Big) \;=\; \frac{1}{(1-x)^t}.$$

We can get several interesting specializations of this bivariate function. First, taking $t=1$ gives us the single-variable exponential generating function:

$$\tilde{P}(x) \;=\; \exp\Big(\log(\tfrac{1}{1-x})\Big) \;=\; \frac{1}{1-x}\,.$$

Indeed, we can realize permutations either as sets of cycles or as labeled sequences: $\tilde{\mathcal{P}} \cong \textsc{Set}(\textsc{Cyc}\widetilde{[1]}) \cong \textsc{Seq}(\widetilde{[1]})$; and the above computation is the generating function version.

Next, fixing $k$ and taking the $t^k$ coefficient gives the single-variable generating function:

$$\tilde{P}^{(k)}(x) \;=\; \sum_{n\geq 0} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} \;=\; \frac{1}{k!}\left(t\log(\tfrac{1}{1-x})\right)^k.$$

This does not give an explicit formula for the Stirling cycle numbers, but it allows complex analytic methods to give the asymptotic approximation: $\left[{n \atop k}\right] \sim \frac{(n-1)!}{(k-1)!}(\log n)^{k-1}$ as $n\to\infty$. This means that for large $n$, the fraction of $n$-permutations which have $k$ cycles is very close to $\frac{(\log n)^{k-1}}{n(k-1)!}$.

Finally, fixing $n$ and taking the coefficient of $\frac{x^n}{n!}$ gives the generating function $P_n(t) = \sum_{k=1}^{n} \left[{n \atop k}\right] t^k$. The Taylor Coefficient Formula gives:

$$P_n(t) \;=\; \frac{d^n}{dx^n}\tilde{P}(x,t)\bigg|_{x=0} \;=\; \frac{d^n}{dx^n}\left(\frac{1}{(1-x)^t}\right)\bigg|_{x=0} \;=\; t(t{+}1)(t{+}2)\cdots(t{+}n{-}1) \;=\; t^{\overline{n}}.$$

That is:

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k \;=\; t^{\overline{n}}.$$

Substituting $-t$ for $t$ and factoring out signs changes rising powers to falling:

$$\sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k \;=\; t(t{-}1)\cdots(t{-}n{+}1) \;=\; t^{\underline{n}}.$$

Compare this with our formula involving Stirling partition numbers:[1]

$$\sum_{k=1}^{n} \left\{ n \atop k \right\} t^{\underline{k}} \;=\; t^n.$$

These formulas mean that $(-1)^{n-k}\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ are the change-of-basis coefficients between the usual power basis $1, t, t^2, t^3, \ldots$ for the polynomials in $t$, and the falling power basis $1, t, t^{\underline{2}}, t^{\underline{3}}, \ldots$. Hence, if for any $N$ we define lower-triangular matrices $M_1 = \left((-1)^{n-k}\left[{n \atop k}\right]\right)_{n,k=1}^{N}$

[1]Bijective proof. The right side $t^n$ counts all functions $f : [n] \to [t]$ for $t \in \mathbb{N}$, each of which can be factored into a surjective function, $\mathrm{surj}(n,k) = k!\left\{{n \atop k}\right\}$; and a choice of image, $\binom{t}{k} = \frac{1}{k!}t^{\underline{k}}$.

and $M_2 = \left(\left\{{n \atop k}\right\}\right)_{n,k=1}^N$, then these are *inverse* matrices: $M_1 M_2 = I = (N{\times}N)$ identity matrix. This is equivalent to the formula:

$$\sum_{j\geq 0}(-1)^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}\begin{Bmatrix} j \\ k \end{Bmatrix} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

Similarly, $M_2 M_1 = I$ is equivalent to:

$$\sum_{j\geq 0}(-1)^{j-k}\begin{Bmatrix} n \\ j \end{Bmatrix}\begin{bmatrix} j \\ k \end{bmatrix} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

These formulas can also be proved by the Involution Principle, the first using the same $\infty/00$ Involution used for $t^{\underline{n}} = \sum_{j=1}^n (-1)^{n-j}\left[{n \atop j}\right] t^j$ on p. 5.

The second can be proved using the Lonely/Crowded Involution. The left side counts permuted set partitions: an unordered partition $S_1 \sqcup \cdots \sqcup S_j = [n]$, numbered so that $\min(S_1) < \cdots < \min(S_j)$, along with a permutation $w \in \mathfrak{S}_j$, and given the sign $(-1)^{j-k}$. The involution will define a new permuted set partition $(S', w')$. Take the smallest $a \in [n]$ such that, for $a \in S_i$, the cycle of sets $S_i \cup S_{w(i)} \cup S_{w(w(i))} \cup \cdots$ contains at least two elements. If $S_i = \{a\}$ is a singleton, then join it with the next set on its cycle: $S'_i = S_i \cup S_{w(i)}$ and $w'(i) = w(w(i))$. If $S_i$ is not a singleton, split it into two sets along the same cycle: $S'_i = \{a\}$ and $S'_{w'(i)} = S_i - \{a\}$, with $w'(w'(i)) = w(i)$. This changes the sign $(-1)^{j-k}$ by incrementing/decrementing $j$ while leaving $n, k$ fixed. If there is no such $a$, then this is the unique fixed point consisting of all singleton sets and $w = \mathrm{id}$. In the signed count $\sum_{j\geq 0}(-1)^{j-k}\left\{{n \atop j}\right\}\left[{j \atop k}\right]$, $I$ pairs up and cancels all terms except the fixed point, which gives the right side.

**Cycle formula.** We give another proof of:

$$t^{\underline{n}} = \sum_{k=1}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix} t^k .$$

For a whole number value $t \in \mathbb{N}$, the left side $t^{\underline{n}}$ can be interpreted as the number of:

- injective functions $f : [n] \to [t]$
- proper $t$-colorings of the complete graph $K_n$
- sequences $(a_1, \ldots, a_n)$ of $a_i \in [t]$ with $a_i \neq a_j$ for $i < j$
- vectors in the hyperplane complement $\mathbb{F}_t^n - \bigcup_{i<j} H_{ij}$, where $\mathbb{F}_t$ is a finite field, and $H_{ij}$ is the hyperplane having the $i, j$ coordinates equal.

We can obtain the right side from the fourth interpretation, by using Mobius inversion (generalized PIE) on the lattice $\mathcal{L}(\mathcal{B})$ of subspaces $V \in \mathbb{F}_t^n$ generated by the braid arrangement $\mathcal{B} = \bigcup_{i<j} H_{ij}$. We give this the partial order of *reverse* inclusion: $U \leq V$ means $U \supset V$, so that the minimal element is the entire space $\hat{0} = \mathbb{F}_t^n$. This poset is isomorphic

to $\Pi_n$, the set partitions $S = \{S_1, \ldots, S_k\}$ with $S_1 \sqcup \cdots \sqcup S_k = [n]$, ordered by refinement. Note that under this isomorphism, $\dim(V)$ is equal to $\ell(S) = k$.

Using the functions $f, g : \mathcal{L}(\mathcal{B}) \to \mathbb{Z}$ given by:

$$f(V) \;=\; \#V = t^{\dim(V)}, \qquad g(V) \;=\; \#(V - \bigcup_{H \not\supset V} H)$$

where the union runs over all hyperplanes $H = H_{ij}$ not containing $V$. Then:

$$f(U) \;=\; \sum_{V \subset U} g(V) \quad \Longleftrightarrow \quad g(U) \;=\; \sum_{V \subset U} \mu(U,V) f(V) = \sum_{V \subset U} \mu(U,V) t^{\dim(W)},$$

where $\mu(W,U)$ is the Mobius function of $\mathcal{L}(\mathcal{B})$ defined by $\mu(U,U) = 1$ and $\sum_{U \leq V \leq W} \mu(U,V) = 0$ for $U < W$. In particular:

$$t^{\underline{n}} \;=\; \#(\mathbb{F}_t^n - \bigcup_{i<j} H_{ij}) \;=\; g(\hat{0}) \;=\; \sum_{V \in \mathcal{L}(\mathcal{B})} \mu(\hat{0}, V)\, t^{\dim(V)} \;=\; \sum_{S \in \Pi_n} \mu(\hat{0}, S)\, t^{\ell(S)}.$$

The above expression is called the *characteristic polyomial* of the subspace arrangement: in fact, the chromatic polynomial of any graph is equal to the characteristic polyomial of the corresponding graphical hyperplane arrangement.

Taking the $t^1$ term of the above expression, corresponding to the maximal element $\hat{1} = V = \mathbb{F}_t(1, \ldots, 1)$, we find that $\mu(\hat{0}, \hat{1}) = \mu(\mathbb{F}_t^n, 0) = (-1)^{n-1}(n-1)!$, and from the product poset structure of the interval $[\hat{0}, S]$ for $S = \{S_1, \ldots, S_k\}$ with $\ell(S) = k$ and $n_i = \#S_i$, we easily find that $\mu(\hat{0}, S.) = (-1)^{n-k} \prod_i (n_i - 1)!$. Therefore:

$$t^{\underline{n}} \;=\; \sum_{S \in \Pi_n} (-1)^{n-\ell(S)} \Big( \prod_{i=1}^{\ell(S)} (n_i - 1)! \Big)\, t^{\ell(S)}\,.$$

Now, we can construct any permutation by partitioning $[n]$ into subsets of size $n_1, \ldots, n_k$, and putting the elements of each block into an $n_i$-cycle in one of $(n_i-1)!$ ways. Thus, for a given $k = \ell(S)$, the above expression counts all permutations with of $[n]$ with $k$ cycles, and we have:

$$t^{\underline{n}} \;=\; \sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

We obtain yet another formula for $t^{\underline{n}}$ from the second and third interpretations above. A conjunction of conditions $a_i = a_j$ can be regarded as a set of pairs $\{i, j\}$ in the powerset $\wp\binom{[n]}{2}$, a Boolean poset. Ordinary PIE corresponding to the poset $\wp\binom{[n]}{2}$ gives:

$$t^{\underline{n}} \;=\; \sum_{G \subset K_n} (-1)^{e(G)} t^{c(G)},$$

where $G$ runs over all graphs on $n$ labeled vertices, with $e(G)$ edges and $c(G)$ connected components. Comparing this to the previous formula, each pair $\{i, j\}$ can be considered as a relation $i \sim j$, and a set of pairs generates an equivalence relation corresponding to a set partition in $\Pi_n$. Note that each $S \in \Pi_n$ corresponds to $c_{n_1} \cdots c_{n_k}$ sets in $\wp\binom{[n]}{2}$, where $c_j$ is the number of connected graphs on $j$ labeled vertices, so there are much fewer terms in the $\Pi_n$ formula, and much fewer still in the $k = 1, \ldots, n$ formula.

**Bijective proof of cycle formulas.** We first prove the positive formula:

$$t^{\overline{n}} \;=\; \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

The left side counts permutation partitions: that is, an ordered set partition of $[n]$ into sets $(S_1, \cdots, S_k)$, where $S_i$ may be empty, along with $(w_1, \ldots, w_k)$, where $w_i \in \mathfrak{S}_{S_i}$ is a permutation of the set $S_i$. (One may picture this as an arrangement of $n$ distinct flags on $t$ flagpoles.) The right side counts pairs $(w, f) \in \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$, where $\mathcal{F} = \{\text{functions } f : [n] \to [t]\}$ and $\mathcal{F}^w$ means the functions invariant under $w$, i.e. $f$ is constant on each cycle of $w$, so that $|\mathcal{F}^w| = t^{\mathrm{cyc}(w)}$.

There is an easy bijection between these objects which proves the formula. Given a permutation partition $(S_1, \ldots, S_k; w_1, \ldots, w_k)$, let $f(j) = i$ for $j \in S_i$, and let $w = w_1 \cdots w_k$. Conversely, given $(w, f)$, let $S_i = f^{-1}(i)$, and let $w_i$ be the permutation $w$ restricted to $S_i$.

We may also consider this as an example of Polya's Method (or Burnside's Lemma) counting orbits of group actions. Dividing both sides by $n!$ gives:

$$\left(\!\!\binom{t}{n}\!\!\right) \;=\; \frac{1}{n!} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

The left side counts multisets with $n$ objects of $t$ kinds, which is the quotient of $\mathcal{F}$ under the natural action of $\mathfrak{S}_n$. Now Burnside's Lemma gives:

$$\left| \frac{\mathcal{F}}{\mathfrak{S}_n} \right| \;=\; \frac{1}{|\mathfrak{S}_n|} \sum_{w \in \mathfrak{S}_n} |\mathcal{F}^w|,$$

where $\mathcal{F}^w$ is the set of functions invariant under $w$. This translates directly to the two sides of the multiset formula.

Finally, we give an involution proof of the signed formula

$$t^{\underline{n}} \;=\; \sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

Now the left side counts injective functions $\mathcal{E} = \{f : [n] \hookrightarrow [t]\}$, while the right side is the signed count of $\coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$, where we define $\mathrm{sgn}(w, f) = (-1)^{n-k}$, where $w$ has $k$ cycles.

Now we define a sign-reversing involution

$$I : \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w \to \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$$

with fixed-point set $(\coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w)^I = (\mathrm{id}, \mathcal{E})$, which will prove the formula. For $(w, f) \in \coprod_{w \in \mathfrak{S}_n} \mathcal{F}^w$ with $f$ injective, let $I(w, f) = (w, f)$; since $f$ is constant on all cycles of $w$, it must have only 1-cycles, so $w = e$ and $(w, f) \in (\mathrm{id}, \mathcal{E})$. If $f$ not injective, suppose $f(a) = f(b)$ for minimal $a, b \in [n]$; then define $I(w, f) = (w(ab), f)$, multiplying $w$ by the transposition $(ab)$, so that $w(ab) = (cd)w$ for $c = w(a)$, $d = w(b)$. This reverses sign, since if $a, b$ lie in the same cycle of $w$, then $w(ab)$ cuts this cycle into two; if $a, b$ lie on different cycles, then $w(ab)$ joins these cycles into one. Thus, $I$ will cancel all non-injective $(w, f) \in \coprod_{w\in\mathfrak{S}_n} \mathcal{F}^w$ on the right side of the formula, leaving only the injective $(\mathrm{id}, \mathcal{E})$ on the left side. We call this the $\infty/00$ Involution because it splits a figure-eight cycle into two cycles, and vice versa: it is useful because it toggles every reasonable definition of sign for $w$, incrementing/decrementing both the number of cycles and the number of transpositions.

This argument may also be seen as an example of counting orbits, this time with signs. Dividing by $n!$ gives:

$$\binom{t}{n} \;=\; \frac{1}{n!}\sum_{k=1}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

The left side counts $n$-element subsets of $[t]$, which is the quotient of $\mathcal{E}$ under the free action of $\mathfrak{S}_n$. Also $\mathcal{E} \cong (\mathrm{id}, \mathcal{E}) = (\coprod_{w\in\mathfrak{S}_n} \mathcal{F}^w)^I$, so by the Involution Principle:

$$\left|\frac{\mathcal{E}}{\mathfrak{S}_n}\right| \;=\; \frac{1}{|\mathfrak{S}_n|}|(\coprod_{w\in\mathfrak{S}_n}\mathcal{F}^w)^I| \;=\; \frac{1}{|\mathfrak{S}_n|}\sum_{w\in\mathfrak{S}_n}(-1)^{n-\mathrm{cyc}(w)}\,|\mathcal{F}^w|,$$

which clearly translates to the signed cycle formula.