Coefficients of Rational Functions. Consider a rational generating function:

$$F(x) \;=\; \frac{P(x)}{Q(x)} \;=\; \sum_{k\geq 0} f(k)x^k \;\in\; \mathbb{C}[[x]],$$
$$Q(x) \;=\; \prod\nolimits_i (1-\gamma_i x)^{d_i} \;=\; 1 + a_1 x + \cdots + a_d x^d,$$

so that $x = \frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \ldots$ are the distinct roots of $Q(x)$. Suppose $P(x)/Q(x)$ has *negative degree*, meaning $\deg P(x) < \deg Q(x)$.

THEOREM 1: (i) There exist $p_i(k)$, polynomials in k of degree less than d_i, so that:

$$f(k) \;=\; \sum\nolimits_i p_i(k)\gamma_i^k \quad \text{for all } k \geq 0,$$

satisfying the recurrence: $f(k) = -a_1 f(k{-}1) - \cdots - a_d f(k{-}d)$ for $k \geq d$.

(ii) If $Q(x) = (1-x)^d$, then $f(k)$ is a polynomial in k of degree at most d (or exactly d if $P(x)/Q(x)$ is in lowest terms).

(iii) If $Q(x) = \prod_j (1-x^{m_j})$, then $f(k)$ is a *quasi-polynomial*, meaning there exist polynomials $f_1(k), \ldots, f_m(k)$ such that $f(k) = f_i(k)$ for all $k \equiv i \mod m$, where $m = \operatorname{lcm}\{m_j\}$

Note: A rational function with higher-degree numerator than denominator can be written as $R(x) + \frac{P(x)}{Q(x)}$, where $R(x)$ is a polynomial and $\frac{P(x)}{Q(x)} = \sum_{k\geq 0} f(k)x^k$ is as in the Theorem. Adding $R(x)$ only changes the first few coefficients $f(k)$, namely $k \leq \deg R(x)$.

Proof of Theorem. (i) The formula for $f(k)$ follows from the partial fraction expansion:

$$\frac{P(x)}{\prod_i (1-\gamma_i x)^{d_i}} \;=\; \sum_i \frac{P_i(x)}{(1-\gamma_i x)^{d_i}},$$

together with the negative binomial series $1/(1{-}x)^d = \sum_{k\geq 0} \left(\!\binom{d}{k}\!\right) x^k$, having coefficient sequence: $\left(\!\binom{d}{k}\!\right) = \binom{d+k-1}{k} = \binom{d+k-1}{d-1} = \frac{1}{d!}(k{+}1)(k{+}2)\cdots(k{+}d{-}1)$, a polynomial in k of degree $d{-}1$. The recurrence follows from $(1 + a_1 x + \cdots + a_d x^d)\sum_{k\geq 0} f(k)x^k = P(x)$, whose x^k-coefficient is zero for $k \geq d$. Now (ii) and (iii) follow, when γ_i's are 1 or roots of unity. □

Diophantine homogeneous linear systems. Encode a system of m homogeneous linear equations in n variables by an $m{\times}n$ integer matrix $\Phi \in M_{m\times n}(\mathbb{Z})$, and consider the set of non-negative integer solutions:

$$\mathcal{E} \;=\; \{\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{N}^n \mid \Phi\boldsymbol{\alpha} = \mathbf{0}\}.$$

This set is a *monoid*, meaning $\mathcal{E}$ is closed under vector addition (but not subtraction).

Denote the vector space of real solutions by $V = \{\mathbf{v} \in \mathbb{R}^n \mid \Phi\mathbf{v} = \mathbf{0}\}$, having dimension $d = n - \operatorname{rank}(\Phi)$, and let $C = V \cap \mathbb{R}^n_{\geq 0}$ be the infinite wedge-shaped region cut out of V by the positive octant in $\mathbb{R}^n$. Thus, $\mathcal{E}$ is the set of $\mathbb{Z}^n$-lattice points in $C \subset V \subset \mathbb{R}^n$.

More generally, a d-dimensional *polyhedral cone* $C \subset V \cong \mathbb{R}^d$ is the intersection of finitely many half-spaces $H^+_{\boldsymbol{\epsilon}} = \{\mathbf{v} \in V \mid \mathbf{v}\cdot\boldsymbol{\epsilon} \geq 0\}$, such that C contains an open subset of V, but C does not contain any line. A *face* of C is the intersection of C with any hyperplane $H_{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}^\perp$, provided $C \subset H^+_{\boldsymbol{\epsilon}}$. Each face is a lower-dimensional cone, including $\{\mathbf{0}\} \subset C$, and we call C itself the unique d-dimensional face. A face of dimension 1 is called an *extremal ray*, and we may write C as the convex span of its extremal rays: $C = \mathbb{R}_{\geq 0}\boldsymbol{\alpha}_1 + \cdots + \mathbb{R}_{\geq 0}\boldsymbol{\alpha}_r$. A d-dimensional *simplicial cone* $\sigma \subset V$ is the intersection of d half-spaces $\sigma = H^+_{\boldsymbol{\epsilon}_1} \cap \cdots \cap H^+_{\boldsymbol{\epsilon}_d}$, where $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_d\}$ is a basis of V; and the extremal rays $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_d\}$ of σ also form a basis.

Combinatorics of triangulation. A *triangulation* of C is a set Γ of simplicial cones which form a conical simplicial complex decomposing C: that is, $C = \bigcup_{\sigma\in\Gamma} \sigma$; any face of $\sigma \in \Gamma$ is also in Γ; and the intersection of any $\sigma, \tau \in \Gamma$ is also in Γ.

We can produce a triangulation using the poset $\mathcal{L}(C)$, the faces of C ordered by inclusion, with minimal element the one-point cone $\hat{0} = \{\mathbf{0}\}$, and atoms given by the extremal rays $R_i = \mathbb{R}_{\geq 0}\boldsymbol{\alpha}_i$ for $i = 1, \ldots, r$. One possibility for Γ is the order complex $\Delta(\mathcal{L}(C))$, but this introduces new extremal rays for simplicial cones, one through the center of each face of C.

To make Γ efficiently small, we want bigger simplicial cones using only the extremal rays $R_1, \ldots, R_r$ of C. For each minimal containment of faces $F \subset F'$, an edge of the Hasse diagram of $\mathcal{L}(C)$, we label it with the smallest i such that $F' = F + R_i$; and for every maximal chain of $\mathcal{L}(C)$ with an increasing sequence of labels $i_1 < \cdots < i_d$, we form the simplicial cone $\sigma = R_{i_1} + \cdots + R_{i_d}$; then we take these as the maximal $\sigma \in \Gamma$.

We decompose the triangulation into boundary and internal faces: $\Gamma = \partial\Gamma \sqcup \Gamma^\circ$. Here $\partial\Gamma$ denotes the cones $\tau \in \Gamma$ inside the boundary of C, for example the extremal rays. Also, Γ° denotes the remaining internal cones, whose interiors lie in the interior of C; this includes every maximal cone of Γ. We also complete the poset with a top element: $\hat{\Gamma} = \Gamma \cup \{\hat{1}\}$.

We will need the Möbius function of $\hat{\Gamma}$, which we compute using the topology of the order complex $\Delta = \Delta(\hat{\Gamma})$. Recall that for an interval $[\sigma,\tau] \subset \hat{\Gamma}$, we extend the chain $\sigma < \tau$ upward and downward by saturated chains to get an associated simplex $\nu_{\sigma\tau} = (0 \lessdot \cdots \lessdot \sigma' \lessdot \sigma < \tau \lessdot \tau' \lessdot \cdots \lessdot \hat{1}) \in \Delta$; then $\mathrm{link}(\nu_{st}) = \Delta(s,t)$, the order complex of the open interval (s,t), and $\mu[\sigma,\tau] = \tilde{\chi}(\mathrm{link}(\nu_{st}))$, the reduced simplicial Euler characteristic. Now, a transverse slice through C is a convex polytope homeomorphic to a $(d{-}1)$-ball, so the poset $\hat{\Gamma}-\{\hat{0},\hat{1}\}$ is isomorphic to a triangulation of this ball, $\hat{\Gamma}$ is a triangulation of the double cone over this ball (still a ball); and the order complex Δ is the barycentric subdivision of this triangulation. Since Δ is a ball (in particular a simplicial manifold with boundary), the link of $\nu_{\sigma\tau}$ is either[1] a homology sphere of dimension $\ell(\sigma,\tau){-}2$ (if $\nu_{\sigma\tau}$ is an internal simplex); or it is a homology ball (if $\nu_{\sigma\tau}$ is a boundary simplex, including $[\sigma,\tau] = [\hat{0},\hat{1}]$). In fact, $\nu_{\sigma\tau}$ is on the boundary of Δ whenever $\sigma \in \partial\Gamma$ and $\tau = \hat{1}$, so that $\nu_{\sigma\tau} = (\hat{0} \lessdot \cdots \lessdot \sigma' \lessdot \sigma < \hat{1})$; thus:

$$\mu[\sigma,\tau] \;=\; \begin{cases} (-1)^{\ell(\sigma,\tau)} & \text{if } \sigma \in \Gamma^\circ \text{ or } \tau < \hat{1} \\ 0 & \text{else.} \end{cases}$$

Here $\ell(\sigma,\tau)$ is the length of a saturated chain from σ to τ in $\hat{\Gamma}$. In terms of the dimensions of simplicial cones $\sigma \subset C$, we thus have: $\mu[\sigma,\hat{1}] = (-1)^{d-\dim(\sigma)+1}$ if $\sigma \in \Gamma^\circ$, else $\mu[\sigma,\hat{1}] = 0$.

Generating functions. We transform the data of $\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathcal{E}$ into a multivariate generating function in $\mathbb{C}[[x_1, \ldots, x_n]]$:

$$E(\mathbf{x}) \;=\; E(x_1, \ldots, x_n) \;=\; \sum_{\boldsymbol{\alpha}\in\mathcal{E}} \mathbf{x}^{\boldsymbol{\alpha}} \;=\; \sum_{\boldsymbol{\alpha}\in\mathcal{E}} x_1^{a_1} \cdots x_n^{a_n}.$$

All coefficients are 0 or 1, so we can determine $\mathcal{E}$ completely from $E(\mathbf{x})$. We can also obtain more useful generating functions of $\mathcal{E}$ by specializing the variables in $E(\mathbf{x})$.

To determine $E(\mathbf{x})$, we use the triangulation Γ obtained above. For each simplicial cone $\sigma \in \Gamma$, let:

$$\mathcal{E}_\sigma \;=\; \sigma \cap \mathbb{Z}^n \;\subset\; C \cap \mathbb{Z}^n \;=\; \mathcal{E}.$$

Normalize the extremal rays $R = \mathbb{R}_{\geq 0}\boldsymbol{\alpha}$ of C so that $\boldsymbol{\alpha}$ is the shortest nonzero vector of R which lies in $\mathcal{E}$; Stanley calls these $\boldsymbol{\alpha}$'s the *completely fundamental set* $\mathrm{CF}(C)$. Rearranging indices, we may write $\sigma = \mathbb{R}_{\geq 0}\boldsymbol{\alpha}_1 + \cdots + \mathbb{R}_{\geq 0}\boldsymbol{\alpha}_{d'}$, where $d' = \dim(\sigma)$, but the inclusion

[1] See R.J. Daverman, *Decompositions of Manifolds* (2007), §II.12.

$\mathcal{E}_\sigma \subset \mathbb{N}\boldsymbol{\alpha}_1 + \cdots + \mathbb{N}\boldsymbol{\alpha}_{d'}$ need not be an equality. We must consider the finite set of lattice points in the parallelopiped at the corner of σ:

$$\mathcal{D}_\sigma \;=\; \{\boldsymbol{\beta} \in \mathcal{E}_\sigma \mid \boldsymbol{\beta} = c_1\boldsymbol{\alpha}_1 + \cdots + c_{d'}\boldsymbol{\alpha}_{d'} \text{ for } 0 \le c_i < 1\},$$

so that $\mathcal{E}_\sigma = \mathcal{D}_\sigma + \mathbb{N}\boldsymbol{\alpha}_1 + \cdots + \mathbb{N}\boldsymbol{\alpha}_{d'}$. Equivalently, we have the rational generating function:

$$E_\sigma(\mathbf{x}) \;=\; \Big(\sum\nolimits_{\boldsymbol{\beta}\in\mathcal{D}_\sigma} \mathbf{x}^{\boldsymbol{\beta}}\Big) \prod_{i=1}^{d'} \frac{1}{1-\mathbf{x}^{\boldsymbol{\alpha}_i}}\,.$$

THEOREM 2: Let Γ be a triangulation of the d-dimensional cone C, so that $\mathcal{E} = \bigcup_{\sigma\in\Gamma} \mathcal{E}_\sigma$, and let Γ° denote the internal simplices of Γ. Then the generating function of $\mathcal{E}$ is:

$$E(\mathbf{x}) \;=\; \sum_{\sigma\in\Gamma^\circ} (-1)^{d-\dim(\sigma)} E_\sigma(\mathbf{x}).$$

Proof. Let σ^0 denote the interior of a cone $\sigma \in \Gamma$, an open subset of σ, and let $\mathcal{E}^\circ_\sigma = \mathcal{E} \cap \sigma^0$. Then we have:

$$\mathcal{E}_\tau = \coprod_{\sigma\le\tau} \mathcal{E}^\circ_\sigma,$$

and this holds also for $\tau = \hat{1} \in \hat{\Gamma}$ if we set $\mathcal{E}_{\hat{1}} = \mathcal{E}$ and $\mathcal{E}^\circ_{\hat{1}} = \emptyset$. Performing Möbius inversion:

$$\begin{aligned} E_\tau(\mathbf{x}) &= \sum\nolimits_{\sigma\le\tau} E^\circ_\sigma(\mathbf{x}), \\ E^\circ_\tau(\mathbf{x}) &= \sum\nolimits_{\sigma\le\tau} \mu[\sigma,\tau] E_\sigma(\mathbf{x}). \end{aligned}$$

In particular, using our evaluation of the Möbius function of $\hat{\Gamma}$, we have:

$$\begin{aligned} 0 \;=\; E^\circ_{\hat{1}}(\mathbf{x}) &= E_{\hat{1}}(\mathbf{x}) + \sum\nolimits_{\sigma<\hat{1}} \mu[\sigma,\hat{1}]\, E_\sigma(\mathbf{x}) \\ &= E(\mathbf{x}) + \sum\nolimits_{\sigma\in\Gamma^\circ} (-1)^{d-\dim(\sigma)+1} E_\sigma(\mathbf{x}). \end{aligned}$$

The desired formula for $E(\mathbf{x})$ follows. □

Note that each d-dimensional cone σ is internal, and contributes a term $+E_\sigma(\mathbf{x})$ to $E(\mathbf{x})$.

Reciprocity. THEOREM 3: Consider the rational power series $F(x) = \frac{P(x)}{Q(x)} = \sum_{k\ge0} f(k)x^k$, where $f(k) = \sum_i p_i(k)\gamma_i^k$ as in Theorem 1. Define $f(k)$ by the same formula for all $k \in \mathbb{Z}$, and let $\overline{F}(x) = \sum_{k\ge1} f(-k)x^k$. Then $F(\frac{1}{x}) = -\overline{F}(x)$, meaning the power series of the rational function $F(\frac{1}{x})$ is $-\overline{F}(x)$.

Proof. Consider the doubly-infinite series $\sum_{k\in\mathbb{Z}} f(k)x^k$ in $\mathbb{C}[[x,x^{-1}]]$, the vector space of formal Laurent series. We can formally multiply these series by polynomials, so we have:

$$Q(x)\sum\nolimits_{k\in\mathbb{Z}} f(k)x^k \;=\; \sum\nolimits_{k\in\mathbb{Z}} g(k)x^k,$$

where $g(k) = \sum_i q_i(k)\gamma_i^k$ for some polynomials $q_i(k)$. Since $Q(x)\sum_{k\ge0} f(k)x^k = P(x)$, we know that $g(k) = 0$ for sufficient large $k \gg 0$; but this can only mean that every $q_i(k)$ is the zero-polynomial, and therefore $g(k) = 0$ for all k. That is, in $\mathbb{C}[[x,x^{-1}]]$, we have:

$$-Q(x)\sum\nolimits_{k\ge1} f(-k)x^{-k} \;=\; Q(x)\sum\nolimits_{k\ge0} f(k)x^k \;=\; P(x).$$

Substituting x^{-1} for x in $\mathbb{C}[[x,x^{-1}]]$, we obtain:

$$-Q(x^{-1})\sum\nolimits_{k\ge1} f(-k)x^k \;=\; P(x^{-1}).$$

This also holds in the ring $L = \mathbb{C}[[x]][x^{-1}] = \{\sum_{k\geq N} a_k x^k \mid N \in \mathbb{Z}\}$, a subspace of $\mathbb{C}[[x, x^{-1}]]$. Indeed, L is a field containing an isomorphic copy of the rational functions $\mathbb{C}(x)$, so we have $-Q(\frac{1}{x})\overline{F}(x) = P(\frac{1}{x})$ in $\mathbb{C}(x)$ as desired. □

As in Theorem 2, consider $\mathcal{E} = C \cap \mathbb{Z}^n$ for a d-dimensional cone $C \subset V$, and define C° to be the interior, an open subset of C and of V; and let $\mathcal{E}^\circ = C^\circ \cap \mathbb{Z}^n$. (Stanley denotes $\mathcal{E}^\circ$ as $\overline{\mathcal{E}}$.) Then we have the following reciprocity relation between generating functions of $\mathcal{E}$ and $\mathcal{E}^\circ$.

Theorem 4: $E(\frac{1}{\mathbf{x}}) = (-1)^d E^\circ(\mathbf{x})$, meaning the power series of the rational function $E(\frac{1}{\mathbf{x}}) = E(\frac{1}{x_1}, \ldots, \frac{1}{x_n})$ is $(-1)^d E^\circ(x_1, \ldots, x_n)$.

Proof. For $d = 1$, $\mathcal{E} = \mathbb{N}$, $\mathcal{E}^\circ = \mathbb{N}_{\geq 1}$, $E(x) = \frac{1}{1-x}$, we have: $E(\frac{1}{x}) = \frac{1}{1-1/x} = \frac{x}{1-x} = -E^\circ(x)$. For $C = \sigma$, a d-dimensional simplicial cone, we have $E_\sigma(\mathbf{x}) = (\sum_{\beta\in\mathcal{D}_\sigma} \mathbf{x}^{\boldsymbol{\beta}}) \prod_{i=1}^d \frac{1}{1-\mathbf{x}^{\boldsymbol{\alpha}_i}}$, and it is easy to see that $E^\circ_\sigma(\mathbf{x}) = (\sum_{\beta\in\overline{\mathcal{D}}_\sigma} \mathbf{x}^{\boldsymbol{\beta}}) \prod_{i=1}^d \frac{1}{1-\mathbf{x}^{\boldsymbol{\alpha}_i}}$, where:

$$\begin{aligned} \mathcal{D}_\sigma &= \{\beta \in \mathcal{E} \mid \beta = c_1\boldsymbol{\alpha}_1 + \cdots + c_d\boldsymbol{\alpha}_d \text{ for } 0 \leq c_i < 1\} \\ \overline{\mathcal{D}}_\sigma &= \{\beta \in \mathcal{E} \mid \beta = c_1\boldsymbol{\alpha}_1 + \cdots + c_d\boldsymbol{\alpha}_d \text{ for } 0 < c_i \leq 1\}. \end{aligned}$$

Thus we compute:

$$\begin{aligned} E_\sigma(\tfrac{1}{\mathbf{x}}) &= \left(\textstyle\sum_{\beta\in\mathcal{D}_\sigma} \mathbf{x}^{-\boldsymbol{\beta}}\right) \prod_{i=1}^d \frac{1}{1-\mathbf{x}^{-\boldsymbol{\alpha}_i}} \\ &= \left(\textstyle\sum_{\beta\in\mathcal{D}_\sigma} \mathbf{x}^{-\boldsymbol{\beta}}\right) \prod_{i=1}^d \frac{-\mathbf{x}^{\boldsymbol{\alpha}_i}}{1-\mathbf{x}^{\boldsymbol{\alpha}_i}} \\ &= (-1)^d \left(\textstyle\sum_{\beta\in\mathcal{D}_\sigma} \mathbf{x}^{\mathbf{A}-\boldsymbol{\beta}}\right) \prod_{i=1}^d \frac{1}{1-\mathbf{x}^{\boldsymbol{\alpha}_i}}, \end{aligned}$$

where $\mathbf{A} = \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_d$.

Now consider the transformation $\phi : V \to V$, $\phi(\mathbf{v}) = \mathbf{A} - \mathbf{v}$. Take the abelian group $\mathbb{Z}\mathcal{E}_\sigma$ generated by the monoid $\mathcal{E}_\sigma$, so that $\mathcal{E}_\sigma = \sigma \cap \mathbb{Z}\mathcal{E}_\sigma$. Also consider the parallelepiped $\pi_\sigma = \{c_1\boldsymbol{\alpha}_1 + \cdots + c_d\boldsymbol{\alpha}_d \text{ for } 0 \leq c_i < 1\}$, so that $\mathcal{D}_\sigma = \pi_\sigma \cap \mathbb{Z}\mathcal{E}_\sigma$ and $\phi(\mathbb{Z}\mathcal{E}_\sigma) = \mathbb{Z}\mathcal{E}_\sigma$. Then:

$$\phi(\mathcal{D}_\sigma) \;=\; \phi(\pi_\sigma) \cap \phi(\mathbb{Z}\mathcal{E}) \;=\; \phi(\pi_\sigma) \cap \mathbb{Z}\mathcal{E} \;=\; \overline{\mathcal{D}}_\sigma.$$

That is, $\overline{\mathcal{D}}_\sigma = \{\mathbf{A} - \boldsymbol{\beta} \mid \beta \in \mathcal{D}_\sigma\}$, so the above formula reduces to:

$$E_\sigma(\tfrac{1}{\mathbf{x}}) \;=\; (-1)^d \left(\textstyle\sum_{\beta\in\overline{\mathcal{D}}_\sigma} \mathbf{x}^{\boldsymbol{\beta}}\right) \prod_{i=1}^d \frac{1}{1-\mathbf{x}^{\boldsymbol{\alpha}_i}} \;=\; E^\circ(\mathbf{x}).$$

Finally, consider a general cone C with triangulation Γ. Then the interior of C is a disjoint union of the interiors σ° for $\sigma \in \Gamma^\circ$, and $\mathcal{E}^\circ_\sigma = \coprod_{\sigma\in\Gamma^\circ} \mathcal{E}^\circ_\sigma$. Then we compute:

$$\begin{aligned} E(\tfrac{1}{\mathbf{x}}) &= \textstyle\sum_{\sigma\in\Gamma^0} (-1)^{d-\dim(\sigma)} E_\sigma(\tfrac{1}{\mathbf{x}}) \;=\; \sum_{\sigma\in\Gamma^0} (-1)^{d-\dim(\sigma)} (-1)^{\dim(\sigma)} E^\circ_\sigma(\mathbf{x}) \\ &= (-1)^d \textstyle\sum_{\sigma\in\Gamma^0} E^\circ_\sigma(\mathbf{x}) \;=\; (-1)^d E^\circ(\mathbf{x}). \end{aligned}$$

□