Let T_n be the number of rooted, unlabelled trees with n vertices: $T_0 = 0$, $T_1 = T_2 = 1$, $T_3 = 2$, $T_4 = 4$. For $n = 4$, the tree can be either: a linear path rooted at an end or an internal vertex; or a star rooted at the center or at a leaf; giving $T_4 = 4$ distinct choices.

The combinatorial specification $\mathcal{T} = [1] \times \textsc{MSet}(\mathcal{T})$ gives the following equation involving the generating function $T(x) = \sum_{n \geq 1} T_n x^n$:

$$T(x) = x \prod_{j \geq 1} \frac{1}{(1 - x^j)^{T_j}} .$$

We will apply logarithmic differentiation to obtain an amazing recurrence for T_n. Writing out the equation as:

$$\sum_{n \geq 0} T_{n+1} x^n = \prod_{j \geq 1} (1 - x^j)^{-T_j} ,$$

we apply the operation $x \frac{d}{dx} \log$ to both sides. On the left side, the identity $x \frac{d}{dx} \log f(x) = x f'(x)/f(x)$ implies:

$$x \frac{d}{dx} \log \sum_{n \geq 0} T_{n+1} x^n = \frac{\sum_{n \geq 1} n T_{n+1} x^n}{\sum_{m \geq 0} T_{m+1} x^m} .$$

On the right side, we use $\log(ab) = \log a + \log b$ and $\log(a^b) = b \log a$ to get:

$$x \frac{d}{dx} \log \prod_{j \geq 1} (1 - x^j)^{-T_j} = \sum_{j \geq 1} -T_j \, x \frac{d}{dx} \log(1 - x^j)$$

$$= \sum_{j \geq 1} T_j \frac{k x^j}{1 - x^j} = \sum_{j \geq 1} \sum_{i \geq 1} j T_j \, x^{ij} = \sum_{k \geq 1} (\sum_{j | k} j T_j) x^k .$$

where in the last equality we substitute $k = ij$, and $j|k$ means j divides k.

Now equating the two sides, clearing the denominator, and collecting x^n terms, we get:

$$\sum_{n \geq 1} n T_{n+1} x^n = \sum_{k \geq 1} (\sum_{j | k} j T_j) x^k \cdot \sum_{m \geq 0} T_{m+1} x^m = \sum_{n \geq 1} (\sum_{k=1}^{n} \sum_{j | k} j T_j T_{n-k+1}) x^n$$

where in the second equality we substitute $n = k + m$, so that $m + 1 = n - k + 1$. We conclude:

$$T_{n+1} = \frac{1}{n} \sum_{k=1}^{n} \sum_{j | k} j T_j T_{n-k+1} ,$$

where the right side involves only $T_1, \ldots, T_n$. This recurrence has no combinatorial explanation, but it is fairly efficient computationally.

EXAMPLE: To compute T_5, we sum over $k = 1, 2, 3, 4$ and j running over all divisors of k: that is, $(j, k) = (1, 1)$, (1,2), (2,2), (1,3), (3,3), (1,4), (2,4), (4,4), so that:

$$\begin{aligned} T_5 &= \tfrac{1}{4}(T_1 T_4 + T_1 T_3 + 2 T_2 T_3 + T_1 T_2 + 3 T_3 T_2 + T_1 T_1 + 2 T_2 T_1 + 4 T_4 T_1) \\ &= \tfrac{1}{4}(4 + 2 + 4 + 1 + 6 + 1 + 2 + 16) = 9 . \end{aligned}$$