Math 880 **Optional Homework 10 - Part 2** **Due Fri Dec 15**

The following problems will count for extra credit.

2. A model for combinatorial reciprocity is the formula relating binomial and multichoose coefficients: considering $\binom{n}{k} = \frac{1}{k!}n^{\underline{k}}$ and $\left(\!\binom{n}{k}\!\right) = \frac{1}{k!}n^{\overline{k}}$ as polynomials in n, we have:

$$\binom{-n}{k} \;=\; (-1)^k \left(\!\!\binom{n}{k}\!\!\right).$$

Derive this from our general reciprocity results.

a. Considering $\binom{n}{k}$ as counting the sets $S = \{i_1 < \cdots < i_k\} \subset [n]$, and similarly for $\left(\!\binom{n}{k}\!\right)$ and multisets $M = \{i_1 \leqslant \cdots \leqslant i_k\}$, transform this data into solutions of a Diophantine homogeneous linear system. That is, explicitly define:

$$\mathcal{E} = \{\alpha{=}(t, a_1, \ldots, a_{k+1}) \in \mathbb{N}^{k+2} \mid \Phi\alpha = 0\}$$

so that the level-set $\mathcal{E}_\ell = \mathcal{E} \cap (t{=}\ell)$ and its interior $\mathcal{E}_\ell^\circ = \mathcal{E}^\circ \cap (t{=}\ell)$ give:

$$\#\mathcal{E}_\ell = \left(\!\!\binom{\ell{+}1}{k}\!\!\right) \quad \text{and} \quad \#\mathcal{E}_\ell^\circ = \binom{\ell{-}1}{k}.$$

Hint: Use a "gap" or "step-length" transform, not multiplicities.

b. Form the generating functions $F(t) = E(t, 1, \ldots, 1)$ and $F^0(t) = E^\circ(t, 1, \ldots, 1)$, and apply Theorems 3 and 4 of the Notes to derive the reciprocity formula.

c. Apply Theorem 2 to explicitly evaluate $E(t, x_1, \ldots, x_{k+1})$ and $F(t)$.

d. Apply the Graded Product Principle to the formula from (c), equating $F(t)$ with a very familiar power series. Compare coefficients to obtain a duality for multichoose numbers analogous to $\binom{n}{k} = \binom{n}{n-k}$.

e. Give a direct combinatorial proof of the duality formula in (d) by comparing the multiplicity transform and the gap transform.

3. Consider the ring of symmetric polynomials $\Lambda_3 = \mathbb{Q}[x_1, x_2, x_3]^{S_3}$, with its bases m_λ (monomial), e_λ (elementary), h_λ (homogeneous), s_λ (Schur), indexed by partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3)$. Recall we defined $s_\lambda = a_{\lambda+\rho}/\Delta$ for $\rho = (2, 1, 0)$, $a_\mu = \sum_w \operatorname{sgn}(w) x_{w(\mu)}$, and $\Delta = \prod_{i<j}(x_i{-}x_j)$.

a. By hand, expand e_λ, h_λ, s_λ in terms of m_λ for the polynomials of degrees $1, 2, 3$. Verify the change-of-basis coefficients denoted by Stanley as $M_{\lambda\mu}$ (0-1 matrices), $N_{\lambda\mu}$ ($\mathbb{N}$-matrices), and $K_{\lambda\mu}$ (semi-standard Young tableaux).

b. The Jacobi-Trudi identities give Schur functions in terms of homogeneous and elementary symmetric functions:

$$s_\lambda \;=\; \det(h_{\lambda_i+j-i})_{i,j=1}^n \;=\; \det(e_{\lambda_i^t+j-i})_{i,j=1}^n,$$

where λ^t denotes the transpose partition, and we count $h_i = e_i = 0$ for $i < 0$ and $h_0 = e_0 = 1$. Work out these identities and verify them for the s_λ from (a).

c. Consider the vector space V of matrices $X \in M_{3\times 3}(\mathbb{C})$ with zero trace: $\mathrm{tr}(X) = 0$. Define an action of $G = \mathrm{GL}_3(\mathbb{C})$ by:

$$\rho(A)X \;=\; \det(A)\, AXA^{-1}.$$

Verify that this is a polynomial representation of dimension 8.

Find the common eignevectors of the diagonal matrices $T = \mathrm{diag}(t_1, t_2, t_3)$, compute the character $\chi_V(t_1, t_2, t_3) = \mathrm{tr}\,\rho(T)$, and identify it as a Schur function. Show directly that V is irreducible, containing no subrepresentations except 0 and itself.

d. Repeat (c) for the representation of $\mathrm{GL}_n(\mathbb{C})$ on traceless $n\times n$ matrices. Particularly, what Schur function gives the character?