Math 880 **Homework 10 - Part 1** **Due Wed Dec 6**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

1. Consider the polyhedron $P \subset \mathbb{R}^2$ which is the convex hull of the vertices:

$$v \;=\; (x_1, x_2) \;=\; (0,0),\, (\tfrac{1}{2}, 0),\, (0, \tfrac{1}{2}),\, (1, \tfrac{3}{2}).$$

Goal: Count integer points $i_P(k) = \#(kP \cap \mathbb{Z}^2)$, where kP denotes k-fold dilation. Start with a picture of P and directly count $i_P(k)$ for $k = 1, 2, 3$.

a. Write P as an intersection of half-spaces defined by inequalities $ax_1 + bx_2 \geqslant c$.

b. Let $\mathbb{R}^3$ have coordinates (t, x_1, x_2) and define $C \subset \mathbb{R}^3$, the cone over P, by modifying the above inequalities to $ax_1 + bx_2 \geqslant ct$. Intersecting C with the hyperplane $(t = k)$ gives: $C \cap H_{t=k} = (k, kP)$.

For each inequality $ax_1 + bx_2 \geqslant ct$ which is not simply $x_i \geqslant 0$, introduce a "slack variable" y_j, and modify the inequality to an equation $-ct + ax_1 + bx_2 = y_j$ together with $y_j \geqslant 0$. Given these m equations in n variables, write the $m \times n$ integer matrix $\Phi \in M_{m\times n}(\mathbb{Z})$ such that:

$$\mathcal{E} \;=\; C \cap \mathbb{Z}^3 \;\cong\; \{\, \alpha \in \mathbb{N}^n \mid \Phi \cdot \alpha = \vec{0}\,\}.$$

c. Draw the Hasse diagram of $\mathcal{L}(C)$, the face poset of C, consisting of the minimum element $\hat{0} = \{\vec{0}\}$, the four extremal rays (edges) $R_1, \ldots, R_4$ corresponding to vertices of P, the four planar sides, and the maximum element $\hat{1} = C$. *Hint:* Picture C in $\mathbb{R}^3$, not in $\mathbb{R}^n$.

Embed $\mathcal{L}(C)$ into the Boolean poset $\wp[n]$: take each face F to the set of $i \in [n]$ such that the i^{th} coordinate of every vector $u \in F$ is non-zero.

d. Apply the algorithm from the Notes to produce a triangulation Γ of C by simplicial cones $\sigma \in \Gamma$. That is, for each minimal containment $F \subset F'$, mark the edge of the diagram with the smallest i such that $F' = F + R_i$, and for each maximal chain $\hat{0} \overset{i_1}{\lessdot} \sigma_1 \overset{i_2}{\lessdot} \sigma_2 \overset{i_3}{\lessdot} \hat{1}$ in $\mathcal{L}(C)$ with a decreasing sequence of labels $i_1 > i_2 > i_3$, let $\sigma = R_{i_1} + R_{i_2} + R_{i_3}$ be a maximal cone in Γ; also include all faces of σ.

Draw the Hasse diagram of $\hat{\Gamma} = \Gamma \cup \{\hat{1}\}$, and compute by hand the Möbius function $\mu[\sigma, \hat{1}]$; verify this agrees with the formula in the Notes. (*Hint:* Use the top-down recurrence for $\mu[\sigma, \hat{1}]$, not bottom-up.) Write the generating function formula $\mathcal{E}(\mathbf{x}) = -\sum_{\sigma \in \Gamma^\circ} \mu[\sigma, \hat{1}]\, \mathcal{E}_\sigma(\mathbf{x})$, where Γ° denotes the internal faces $\sigma \in \Gamma$ (whose interiors lie in the interior of C). *Hint:* There are 3 internal faces.

e. Now work out $E_\sigma(\mathbf{x})$. Find the shortest integer vector $\alpha_i = (q, qv_i)$ on each ray R_i. For each cone $\sigma \in \Gamma^\circ$, with extremal rays $\alpha, \alpha', \ldots$, determine $D_\sigma = \{\beta \in \sigma \cap \mathbb{Z}^3 \mid \beta = c\alpha + c'\alpha' + \cdots \text{ for } 0 \leqslant c, c', \ldots < 1\}$. Explicitly compute $E(\mathbf{x})$ using Theorem 2 of the Notes.

f. Define $F(t) = E(t, 1, 1, \ldots)$, and explain why $F(t) = \sum_{k \geqslant 0} i_P(k) t^k$. Find $F(t)$ explicitly as a rational function. Give a partial fraction decomposition, and deduce a formula for $i_P(k)$ as a quasi-polynomial: i.e. find the polynomials $f_1(k), \ldots, f_r(k)$ with $i_P(k) = f_j(k)$ for $k \equiv j \mod r$.

How does the degree of $f_j(k)$ relate to the geometry of P? What about the leading coefficient and the other coefficients? (To read more about this, look up *Pick's Theorem.*)